From Proportions to Linear Relationships

Where might you see the sign shown? What can you say about the triangle on the sign? What do you think 8% represents?

Post-Secondary Proportions

Representations of Proportional Relationships

WARM UP

Determine each equivalent ratio.

1. $\dfrac{7}{16} = \dfrac{x}{48}$

2. $\dfrac{t}{90} = \dfrac{5}{9}$

3. $\dfrac{10}{P} = 1$

4. $250 = \dfrac{1000}{q}$

LEARNING GOALS

- Represent proportional relationships with tables, lines, and linear equations.
- Compare graphs of proportional relationships.
- Compare two different proportional relationships represented in multiple ways.

KEY TERMS

- proportional relationship
- constant of proportionality

You have studied proportional relationships in previous courses. How can you represent and compare proportional relationships using graphs, tables, and equations?

Ratio of Women to Men

Government agencies and civil rights groups monitor enrollment data at universities to ensure that different groups are fully represented. One study focused on the enrollment of women at a certain university. The study found that three out of every five students enrolled were women.

Use the findings of the study to write each ratio.

1. the number of enrolled female students to the total number of students

2. the number of enrolled male students to the total number of students

3. the number of enrolled female students to the number of enrolled male students

4. the number of enrolled male students to the number of enrolled female students

Use the findings of the enrollment study to make predictions.

1. **Determine the number of enrolled female students for each given total number of enrolled students. Explain your reasoning.**

    a. **15 total students**

    b. **250 total students**

   c. **4000 total students**

2. **Compare the total number of enrolled students to the number of enrolled male students.**

    a. **Complete the table.**

Total Students Enrolled in a University	Male Students Enrolled in a University
0	
250	
6000	
	6000

    b. **Explain how you calculated each value.**

3. **Determine the number of female students if 800 enrolled students are male. Show all work and explain your reasoning.**

Does this represent a proportional relationship?
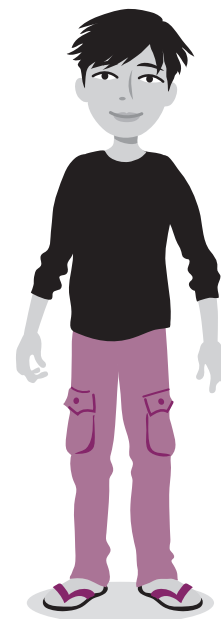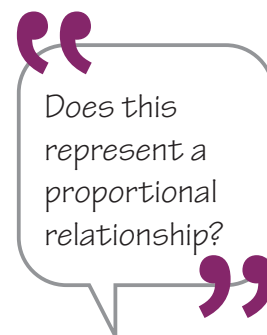
4. **Choose the correct equation to match each description. Then compare the equations.**

$y = \frac{2}{5}x$ $y = 2x + 3$ $y = \frac{2}{3}x$

$y = \frac{3}{2}x$ $y = \frac{3}{5}x$

$y = \frac{5}{2}x$ $y = 2x + 5$ $y = 3x + 2$

a. the number of female students enrolled, y, for x total number of students enrolled

b. the number of male students enrolled, y, for x total number of students enrolled

c. the number of female students enrolled, y, for x male students enrolled

d. the number of male students enrolled, y, for x female students enrolled

e. Describe the similarities and differences in each of the correct equations.

5. Create graphs that display each ratio. Then compare the graphs.

a. the total number of female students enrolled, y, with respect to the total number of students enrolled, x
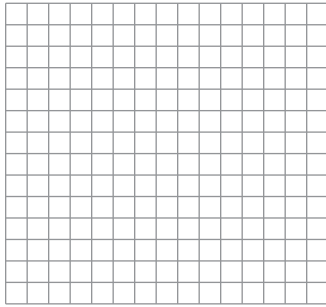
b. the total number of male students enrolled, y, with respect to the total number of students enrolled, x
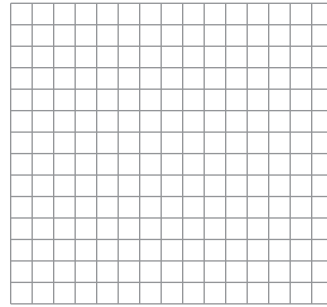
c. Describe the similarities and differences of the two graphs.

In this lesson, you are studying relationships that are proportional. A **proportional relationship** is one in which the ratio of the inputs to the outputs is constant. For example, the ratio of women to men at a university is 3 : 2. Proportional relationships are always written in the form $y = kx$, where x represents an input value, y represents an output value, and k represents some constant that is not equal to 0. The constant k is called the **constant of proportionality**.

6. Identify the constant of proportionality for each relationship in Question 4.

> You will sometimes hear a proportional relationship referred to as a direct variation.

7. Identify the constant of proportionality, or rate of change, for each graph in Question 5. Then explain how to determine k from a graph.
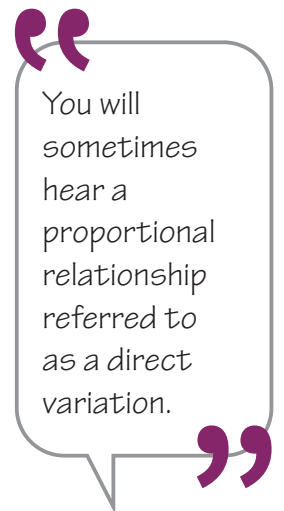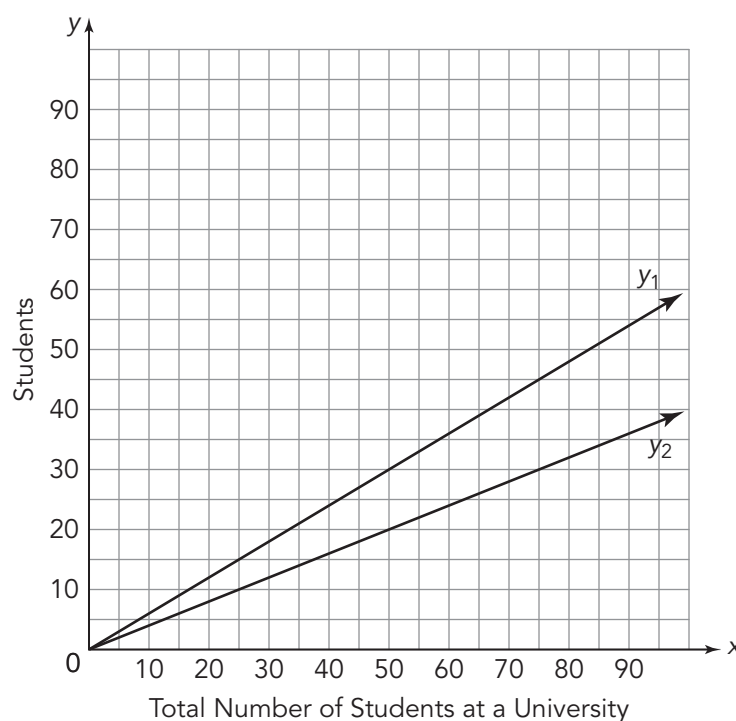
Comparing Ratios and Graphs

Graphs provide a variety of information about relationships between quantities.

> *Can you determine proportionality or dependence?*

1. **Examine the lines graphed on the coordinate plane. What can you determine about the relationships between the quantities by inspecting the graph?**

The lines y_1 and y_2 each represent a proportional relationship. One line represents the proportional relationship between the number of females enrolled and the total number of students. The other line represents the proportional relationship between the number of males enrolled and the total number of students.

2. Determine which line represents each relationship. Explain your reasoning.

   a. the number of females enrolled in a university

   b. the number of males enrolled in a university

The ratio of the number of students who enjoy music to the total number of students is slightly more than the ratio of female students to the total number of students.

3. Draw a line on the coordinate plane that might represent the ratio of the number of students who enjoy music to the total number of students. Label this line y_3. Explain your reasoning.

In a linear relationship, any change in an independent variable will produce a corresponding change in the dependent variable.

The ratio of students who work full-time to total students is less than the ratio of male students to total students.

4. Draw a line on the coordinate plane that might represent the ratio of students at a university who work full-time to the total number of students. Label this line y_4. Explain your reasoning.

Must the lines pass through (0, 0)?
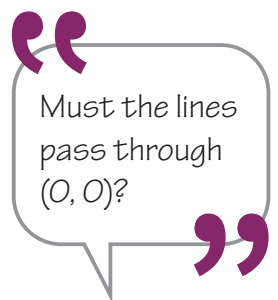
5. Of the lines on the coordinate plane, which is the steepest? How does this relate to the ratios?

Daisa attends college in another state. During summer break, she drives home from college to visit her family and friends.

Daisa's Drive Home

Time (hours)	Distance (miles)
3	180
2	120
1.5	90
2.5	150

1. **Daisa decides to keep track of the time it takes her to drive home from school. She records her distance after various numbers of hours. Her data are shown in the table.**

   a. **Does this table represent a proportional relationship? Explain your reasoning.**

   b. **Write a ratio for distance to time.**

   c. **Write the unit rate for distance per 1 hour.**

Unit rate is a comparison of two quantities in which the denominator has a value of one unit.

One of Daisa's high school classmates, Tymar, attends college with Daisa. He also drives home during the summer break but takes a different route.

2. Analyze the graph of his trip.

    a. Does the graph represent a proportional relationship? Explain your reasoning.

    b. Who drives faster—Daisa or Tymar? Explain your reasoning.

Tymar's Drive Home
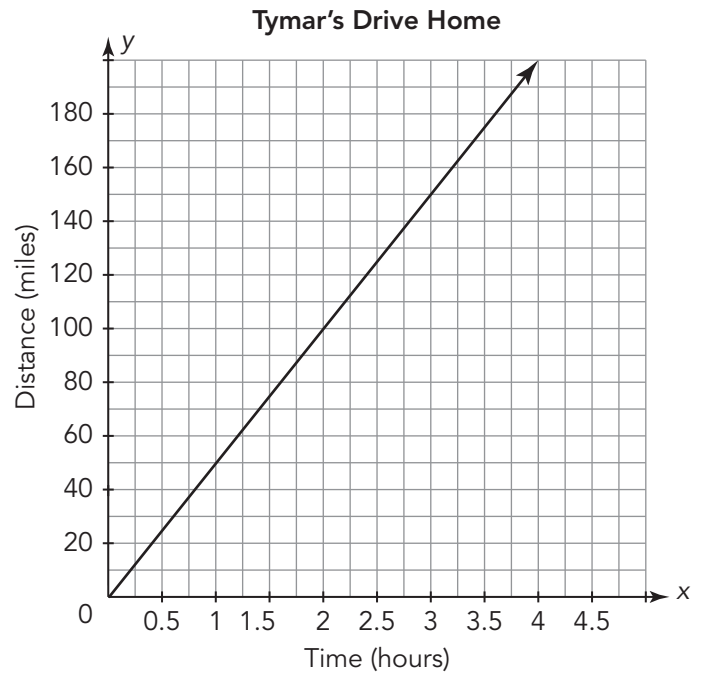
A third friend, Alisha, offers to drive Daisa and Tymar home for spring break so that they can share the cost of gas money. When asked how fast she drives, Alisha reported that the distance traveled, y, for the time, x, can be expressed as $y = 57x$.

3. Does Alisha's equation represent a proportional relationship? Explain your reasoning.

4. Compare the representations of the three friends.

    a. Who drives the fastest? Explain your reasoning.

    b. Rank the friends in order from the slowest driver to the fastest driver.

Comparing Depth of Color

Students in a sculpting class at a university are working in teams to create modeling clay. The students learned that they can make different types of clay by changing the ratio of flour to water. Their recipes are shown in the table.

	Group 1	Group 2	Group 3	Group 4	Group 5	Group 6	Group 7
Flour	2.5 cups	3 cups	7.5 cups	4 cups	12 cups	3.75 cups	5 cups
Water	1 cup	2 cups	3 cups	2 cups	8 cups	1.5 cups	2 cups

1. How many different recipes for clay did the students create? Show all work and explain your reasoning.

The art professor would like all of the projects to include the same shade of orange. The students have learned that orange paint is created by mixing red and yellow paints. Three groups presented suggestions for the shade of orange to be used for the art projects.

Avi's Group	Zander's Group	Paul's Group
$y = \frac{4}{5}x$, where x is the amount of red paint and y is the amount of yellow paint	<table><tr><th>Red Paint (parts)</th><th>Yellow Paint (parts)</th></tr><tr><td>6</td><td>1.5</td></tr><tr><td>8</td><td>2</td></tr><tr><td>12</td><td>3</td></tr><tr><td>15</td><td>3.75</td></tr></table>	

Orange Recipe — graph with x-axis "Red Paint (parts)" and y-axis "Yellow Paint (parts)"

2. **Explain how you know that each group's proposal represents a proportional relationship.**

    a.   **Avi's Group**

    b.   **Zander's Group**

    c.   **Paul's Group**

The greater the ratio of yellow to red paint used, the lighter the shade of orange paint.

3. **Rate the group's proposals from lightest orange to deepest orange. Explain your reasoning.**

4. **Write an equation, where *x* is the amount of red paint and *y* is the amount of yellow paint, that would create a shade of orange that is between the two deepest shades. Explain your reasoning.**
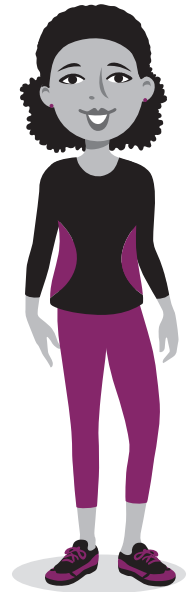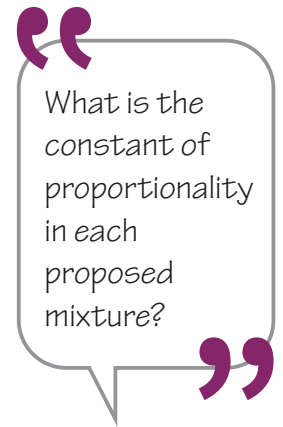
TALK the TALK

Proportional Relationships

All of the relationships in this lesson are examples of proportional relationships.

1. **Complete the graphic organizer to summarize proportional relationships. Include characteristics, examples, and non-examples using tables, equations, and graphs.**

Definition

Characteristics

PROPORTIONAL RELATIONSHIPS

Examples

Non-Examples
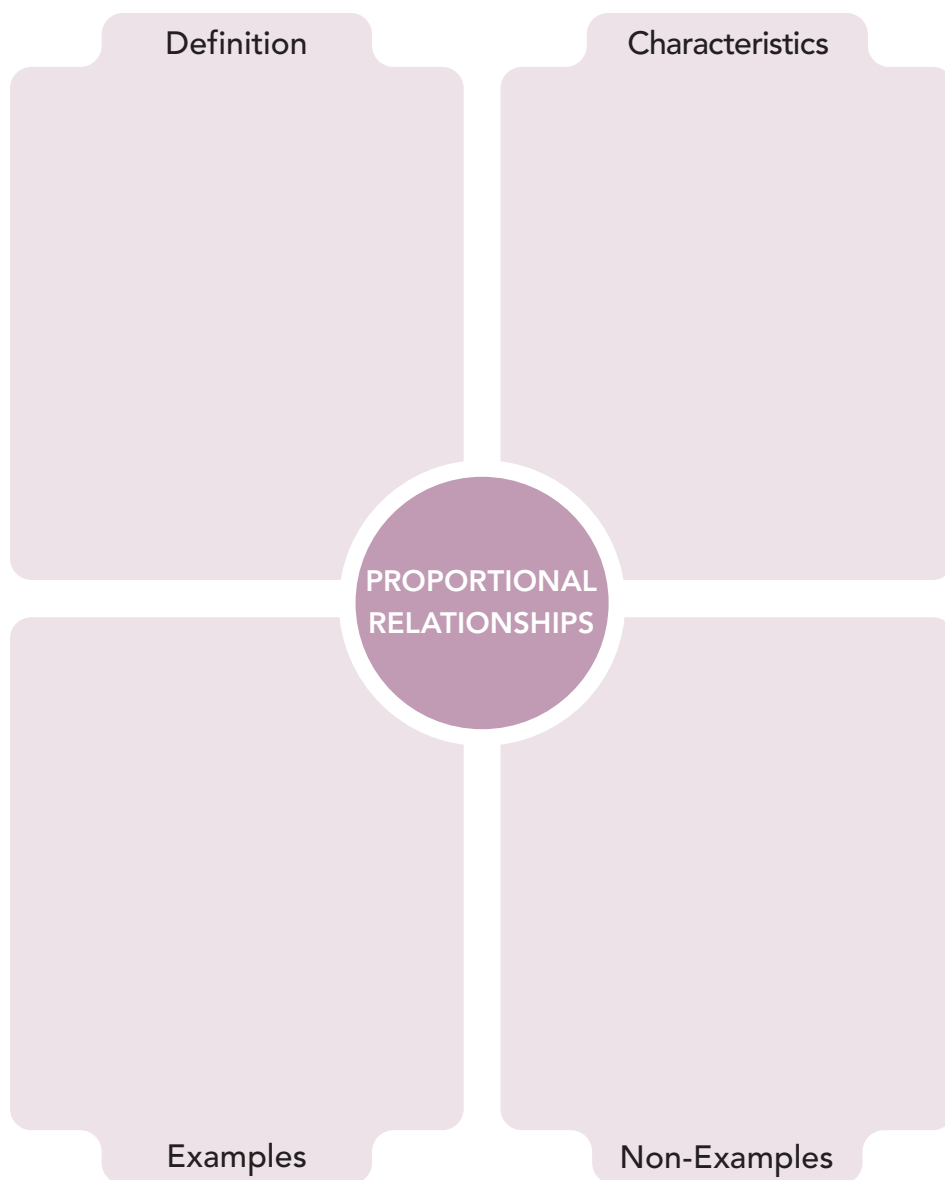
Assignment

Write

Explain how to compare proportional relationships represented in different forms.

Remember

Proportional relationships can be represented using tables, graphs, and equations. In a table, all the ratios of corresponding x- and y-values must be constant. On a graph, a proportional relationship is represented as a linear graph passing through the origin. The equation for a proportional relationship is written in the form $y = kx$, where k is the constant of proportionality.

Practice

1. Determine the constant of proportionality represented in each graph.

a.

b.

c.
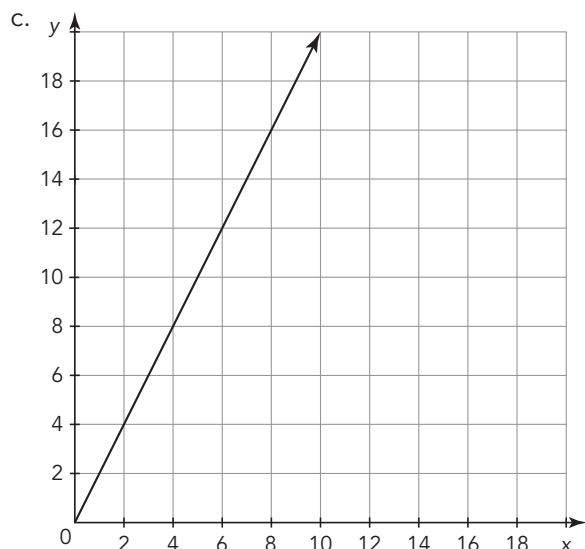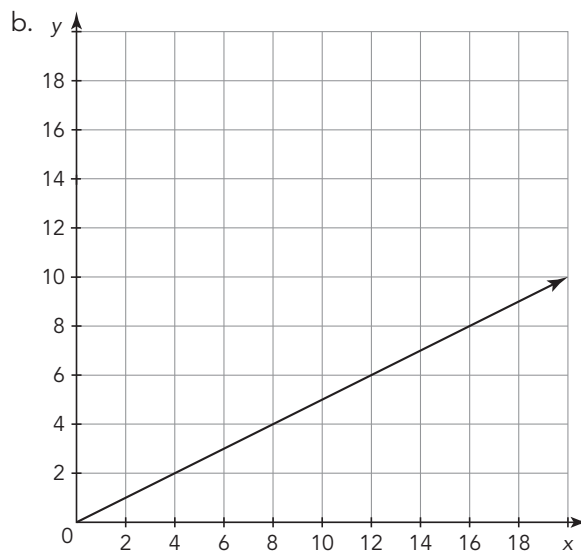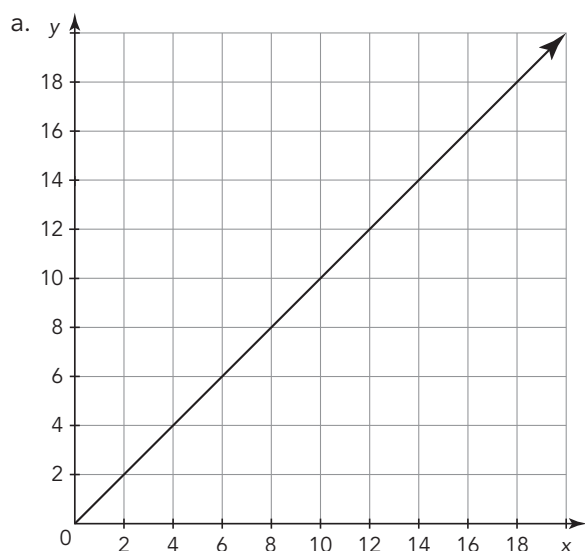
2. Determine the constant of proportionality for each proportional relationship. Assume that y represents all of the outputs and x represents all of the inputs.

a. $2x = 10y$

b. $\left(\frac{3}{5}\right)y = 8x$

c. $\frac{y}{10} = 10x$

d. $\left(\frac{1}{2}\right)x = y$

3. Melanie collects coins from all over the world. She is reorganizing her collection into coins from Europe and coins from other parts of the world. After sorting the coins, she comes to the conclusion that six out of every ten of the coins in her collection come from Europe.

a. Write a ratio for the number of European coins to the total number of coins, the number of non-European coins to the total number of coins, and the number of European coins to the number of non-European coins.

b. Melanie has 230 coins in her collection. Determine the number of European and non-European coins that she has in her collection.

c. Melanie adds to her collection while keeping the same ratio of coins and now has 180 European coins. Determine the number of non-European coins and the total number of coins in her collection.

d. Write an equation to determine the number of European coins, E, if Melanie has t total coins. Show your work and identify the constant of proportionality.

e. Write an equation to determine the number of non-European coins, N, if Melanie has t total coins. Show your work and identify the constant of proportionality.

f. Graph your equations from parts (d) and (e) on a coordinate plane. Label the axes of each graph.

4. Three competing toy stores review their inventory. FunTimeToys creates a graph to represent the relationship between the total number of toys sold and the number of stuffed animals sold. Toy Soldiers writes an equation and The Toy Box creates a table to represent the same information.


FunTimeToys
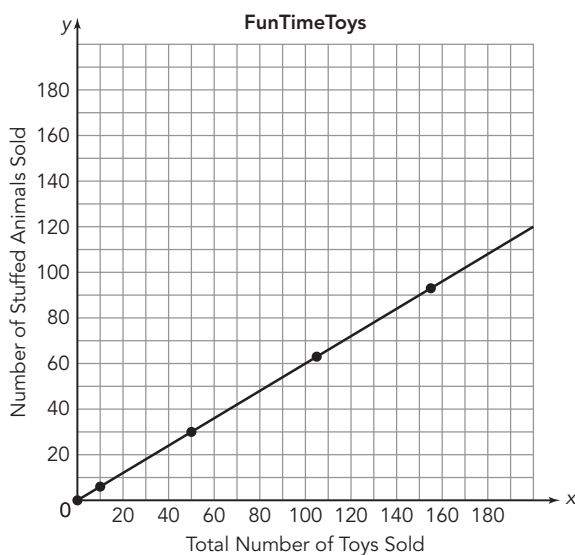
Toy Soldiers $y = \frac{1}{2}x$

The Toy Box

| Total Number of Toys Sold	Number of Stuffed Animals Sold
0	0
12	8
54	36
102	68
156	104

Fluffy Stuffy Stuffed Animals wants to sell their stuffed animals in a local toy store. In which store should they sell their products if they hope to make the most money? Explain your reasoning.

5. Analyze each scenario and graph.

A voice instructor notices that only one out of every ten of her students can sing soprano.

Total Number of Students
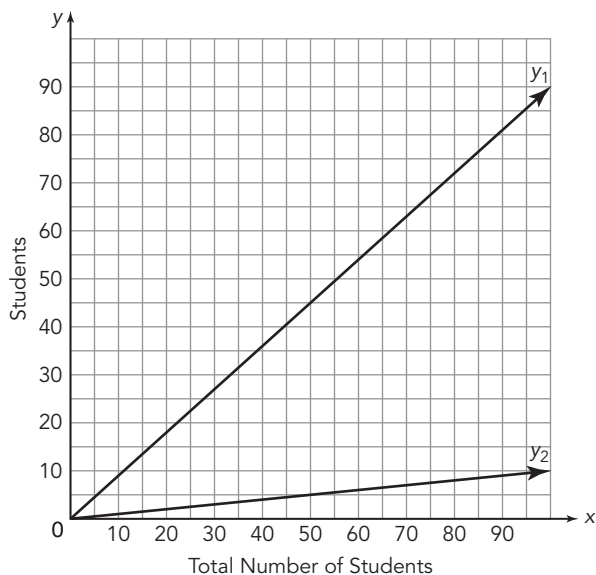
A store owner notices that in his parking lot, two out of every six vehicles are trucks.


Total Number of Vehicles
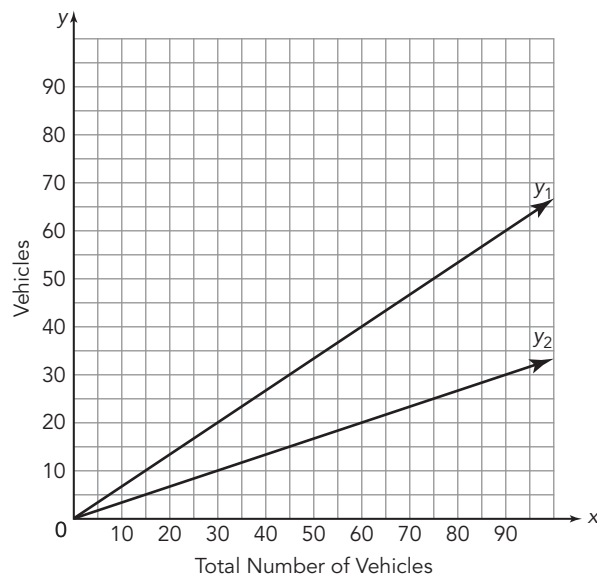
a. Identify the proportional relationship represented by each line as it relates to the scenario. Explain your reasoning.

b. Write an equation that has a constant of proportionality between those represented on the graph. Explain what relationship is represented by your equation.

Stretch

Consider the relationship between the side length of a square and the area of the square. Does this represent a proportional relationship? Use a table of values, equation, and graph to justify your answer.
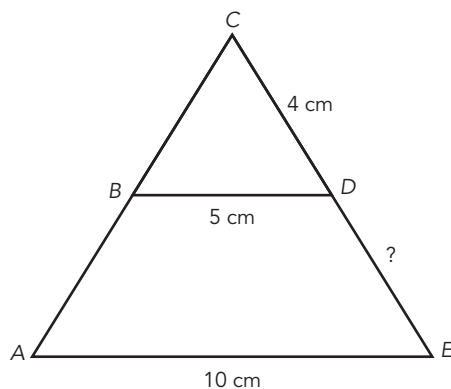
Review

1. In the diagram, $\triangle ABC \sim \triangle XYZ$. State the corresponding sides and angles.
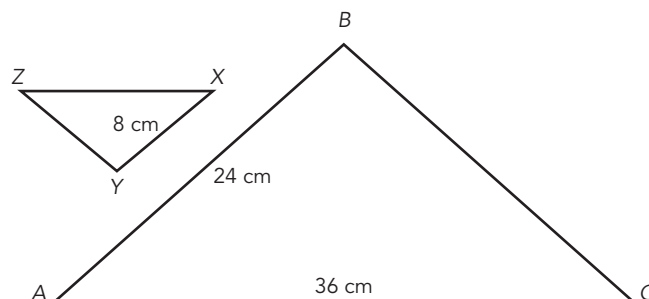
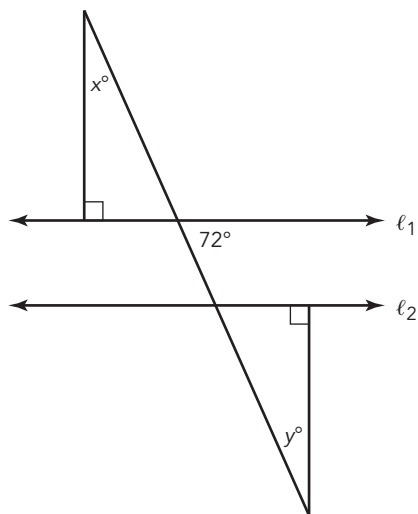2. In the diagram, $\overline{BD} \parallel \overline{AE}$.
   a. Explain why $\triangle BDC \sim \triangle AEC$.
   b. Determine the length of $\overline{DE}$.
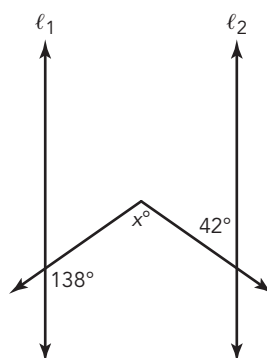

C

4 cm

B   D

5 cm

?

A   E

10 cm

3. Solve for each unknown angle measure given that $\ell_1 \parallel \ell_2$.

a.

$x°$

$72°$

ℓ_1

ℓ_2

$y°$

b.

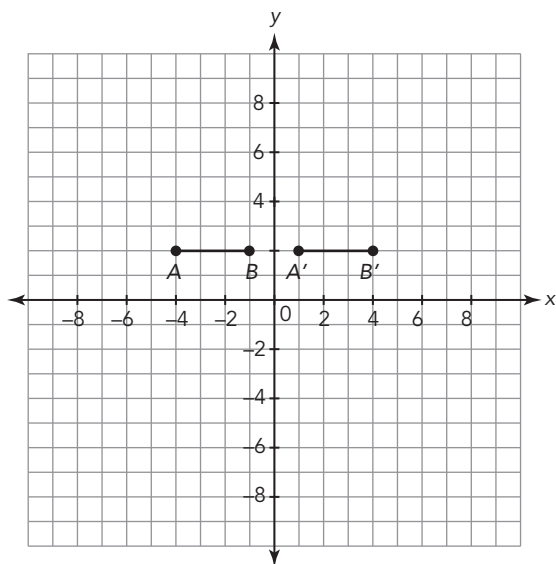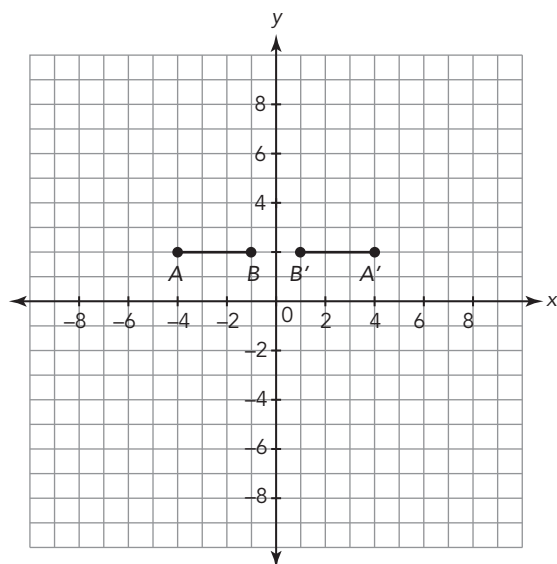ℓ_1 ℓ_2

$x°$ $42°$

$138°$

4. Describe a transformation or sequence of transformations to generate line segment $A'B'$ from original line segment AB.

a.

b.

Jack and Jill Went Up the Hill

2

Using Similar Triangles to Describe the Steepness of a Line

WARM UP

Identify the coefficients and constants in each equation.

1. $64x + 24$

2. $36 - 8z$

3. $-3a^2 + 18a$

4. $42mn + 27m - 1$

LEARNING GOALS

- Analyze the rate of change between any two points on a line.
- Use similar triangles to explore the steepness of a line.
- Derive the equations $y = mx$ and $y = mx + b$, representing linear relationships.
- Graph proportional relationships, interpreting the unit rate as the slope of the graph.

KEY TERMS

- rate of change
- slope

You have learned about rates, unit rates, and the constant of proportionality. How can you connect all of those concepts to describe the steepness of a line?

Let It Steep

Examine each triangle shown.

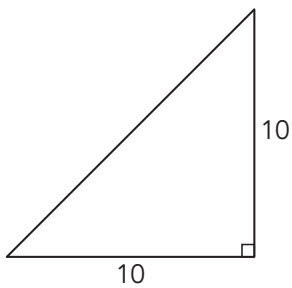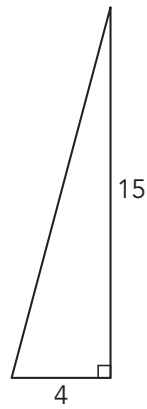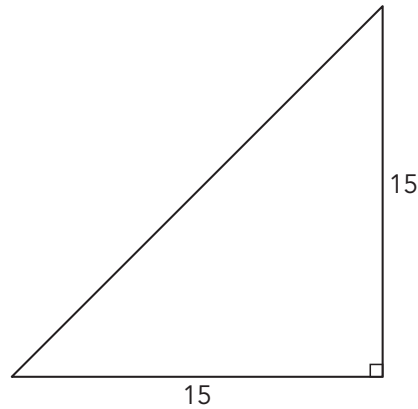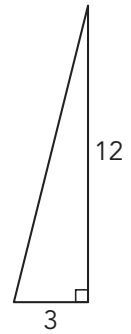| Figure A | Figure B | Figure C | Figure D |

1. **For each triangle, write a ratio that represents the relationship between the height and the base of each triangle.**

2. **Write each ratio as a unit rate.**

3. **How can you use these rates to compare the steepness of the triangles?**

Constant of Proportionality as Rate of Change

On Monday, Jack and Jill walked from their home up a hill to get to the bus stop. They walked 4 yards every 3 seconds.

1. **Write an equation to represent the distance, *d*, Jack and Jill walked over time, *t*.**

2. **Does this situation represent a proportional relationship? If so, identify the constant of proportionality.**

3. **Complete the table. Then graph the points. Finally, draw a line to represent the relationship between the time Jack and Jill walked and their distance from home.**

Time Spent Walking (seconds)	Distance from Home (yards)
	0
1	
3	
	8
7.5	
9	
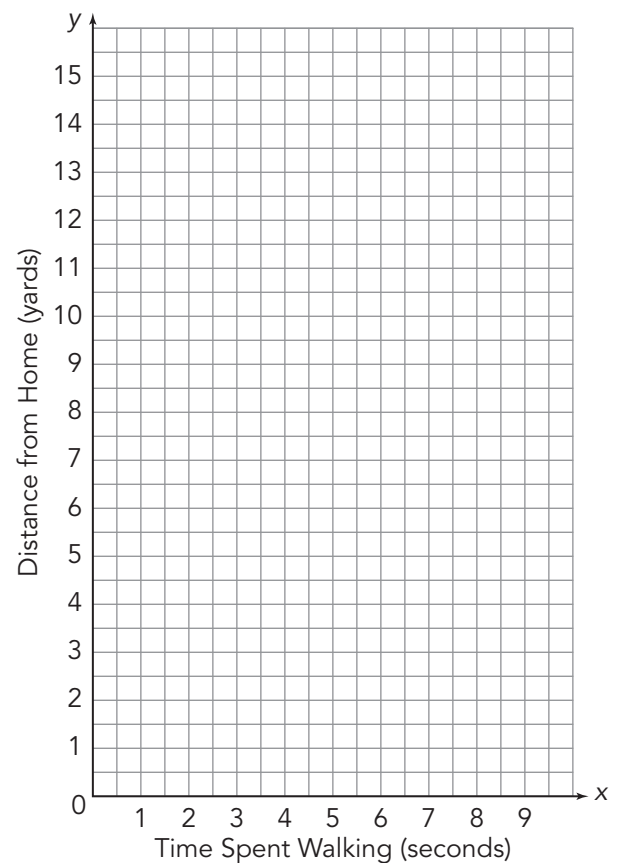
4. What is the unit rate? Explain what the unit rate means in terms of this situation.

5. Explain why Tanner's reasoning is incorrect. Then explain why the graph goes up as you move from left to right.

Tanner 👎

This graph goes up from left to right because Jack and Jill were walking up a hill.

The **rate of change** for a situation describes the amount that the dependent variable changes compared with amount that the independent variable changes.

6. **Consider the Jack and Jill situation.**

   a. **Identify the independent and dependent variables. Explain your reasoning.**

   b. **Identify the rate of change.**

7. **Consider the rate of change, the constant of proportionality, and the unit rate for this situation. What do you notice?**

8. **How would the rate of change and the graph of the relationship change if Jack and Jill walked faster? How would they change if Jack and Jill walked more slowly?**

Distance from Jack and Jill's House (yards)

Time Spent Walking (seconds)
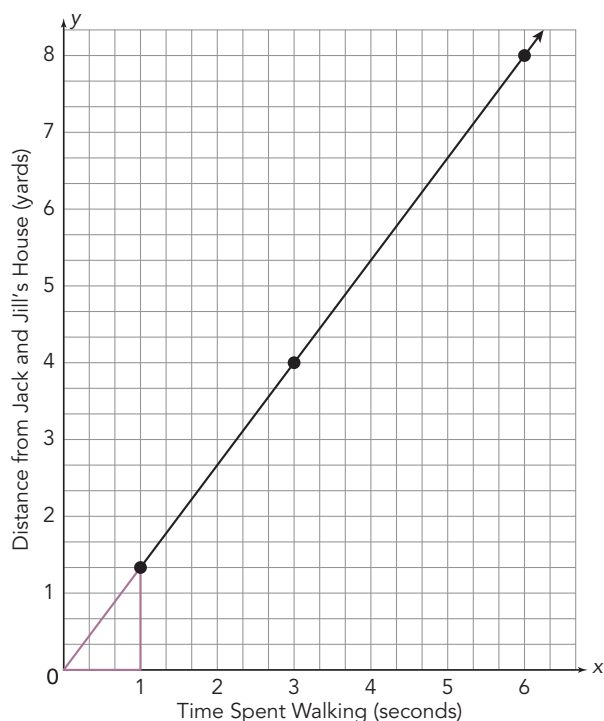
The graph shown represents the relationship between the time Jack and Jill walk and the distance they walk from their home.

Let's analyze three different moments in time during Jack and Jill's walk to the bus stop.

$$t = 1 \quad t = 3 \quad t = 6$$

The graph shows a right triangle drawn to represent $t = 1$.

1. **Trace the triangle on a piece of patty paper. Label the horizontal and vertical sides of the right triangle with their respective lengths.**

2. **Draw right triangles to model $t = 3$ and $t = 6$ on the coordinate plane. Then trace each triangle on a separate piece of patty paper. Label the horizontal and vertical sides of the right triangle with their respective lengths.**

3. **Determine the steepness of each triangle by writing a ratio of the vertical side length to the horizontal side length. How do these ratios compare?**

4. What is the relationship among the three right triangles? Justify your reasoning.

5. Identify and label the triangle that represents the unit rate. Explain how you know.

6. Slide the unit rate triangle along the graph of the line. What do you notice?

7. Slide the other two triangles along the graph of the line. What do you notice?

Keep your patty paper drawings. You will need those in the next lesson.

In the last two activities you investigated a relationship using a rate of change of $\frac{4}{3}$ to represent Jack and Jill walking 4 yards away from their home in 3 seconds, or as a unit rate of $\frac{4}{3}$ yards per second. Because this situation is a proportional relationship, the rate of change can specifically be called the constant of proportionality, represented by the variable k.

In this activity, you created three similar triangles each using two points from the line to explore the steepness of the line. By sliding the similar triangles along the line you noticed the steepness of the line remained constant between any two points on the line. In any linear relationship, **slope** describes the direction and steepness of a line and is usually represented by the variable m. Slope is another name for rate of change. It represents the ratio of the change in vertical distance to the change in horizontal distance between any two points on the line. The slope of a line is constant between any two points on the line.

The sign of the slope indicates the direction of a line. If the slope of a line is positive, then the graph will increase from left to right. If the slope of a line is negative, then the graph will decrease from left to right.

You wrote the equation $d = \frac{4}{3}t$ to represent the distance, d, Jack and Jill walked from home with respect to time, t. Let's generalize this linear relationship.

8. Let *y* represent the dependent variable, *x* represent the independent variable, and *m* represent the slope of the line.

   a. Write a general equation to relate these quantities.

   b. How is this equation similar to the equation for the constant of proportionality?

Equation for a Line Not Through the Origin

Distance from Jack and Jill's House (yards)

Time Spent Walking (seconds)
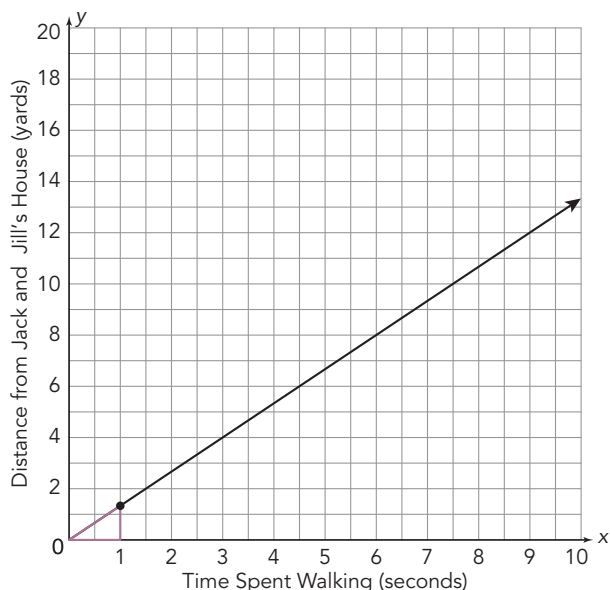
Jack and Jill's Aunt Mary lives 10 yards from their home closer to the bus stop. After spending Monday night at Aunt Mary's house, they leave for the bus stop from there Tuesday morning. They walk at the same rate from either house, 4 yards every 3 seconds.

The graph shows the line $y = \frac{4}{3}x$, which represents the relationship between the time Jack and Jill walk and their distance from their house.

1. Compare the two situations.

a. How do the slopes compare?

b. How do the starting points compare?

2. Let's graph the line to represent their walk to the bus stop from Aunt Mary's house.

a. On a piece of patty paper, trace the line $y = \frac{4}{3}x$ that represents Jack and Jill's walk to the bus stop from their house. Be sure to include the triangle representing the unit rate in your trace.

b. Translate this line to represent their walk from Aunt Mary's house and then transfer this line onto the graph.

3. Analyze the translated line.

a. Does your new line represent a proportional or non-proportional relationship? Explain how you know.

b. How does this translation affect the coordinates of the line? Complete the table to show how the translation affects the coordinates of your new line.

Time Spent Walking (seconds)	Distance from Jack and Jill's House on Monday (yards)	Distance from Jack and Jill's House on Tuesday (yards)
x	y_1	y_2
0	0	
1	$\frac{4}{3}$	
3	4	
6	8	
7.5	10	
9	12	

c. How does this translation affect the unit rate?

d. Write an equation to represent the translated line. Let y_2 represent the distance from Jack and Jill's house and let x represent their time spent walking. Explain how this line is the same and different from the line $y_1 = \frac{4}{3}x$.
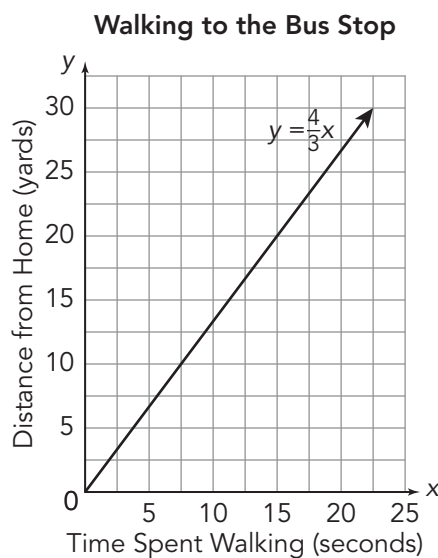
You have written a general equation, $y = mx$, to relate the independent and dependent variables and the slope in a proportional linear relationship. How does this general equation change when the line is translated vertically by b units?

4. Write a general equation to represent the relationship $y = mx$ after it is vertically translated b units.

ACTIVITY
2.4 A Negative Unit Rate

Jack and Jill are walking back home from the bus stop which is 30 yards from their house. They walk at the same rate, 4 yards every 3 seconds.

Consider the two graphs shown.

Walking to the Bus Stop

$y = \frac{4}{3}x$

Distance from Home (yards)

Time Spent Walking (seconds)

Walking Home from the Bus Stop

Distance from Home (yards)

Time Spent Walking (seconds)
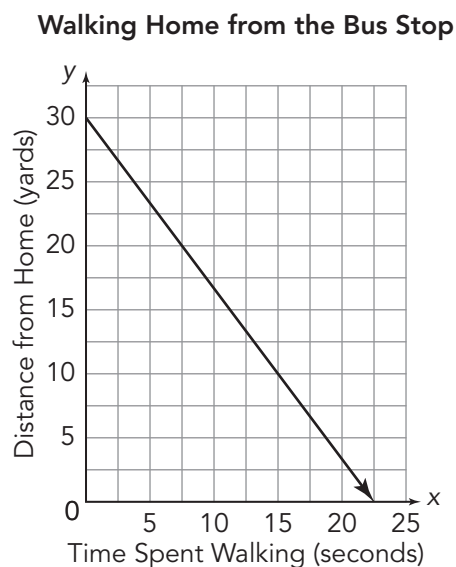
1. Analyze the graph of Jack and Jill walking home from the bus stop.

   a. Does this situation represent a proportional or non-proportional relationship? Explain your reasoning.

   b. Is the slope of the line positive or negative? Explain how you know.

2. Compare and contrast the rate of change, or slope, of each line.

   a. Use patty paper to trace and create any right triangle that represents the rate of change, or slope, from the Walking to the Bus Stop graph.

   b. Place your patty paper on the Walking Home from the Bus Stop graph. How can you transform the right triangle you drew from the Walking to the Bus Stop graph to the Walking Home from the Bus Stop graph?

   c. Slide the right triangle along the line of the Walking Home from the Bus Stop graph. What do you notice?

   d. What is the slope of line in the Walking Home from the Bus Stop graph? Explain your reasoning.

> Remember the slope of a line represents steepness and direction.

3. Write an equation to represent Jack and Jill's walk home from the bus stop. Let *y* represent the distance from home and *x* represent the time spent walking.

4. How does the equation you wrote to represent Jack and Jill's walk home from the bus stop compare to the equation that represents their walk to the bus stop?
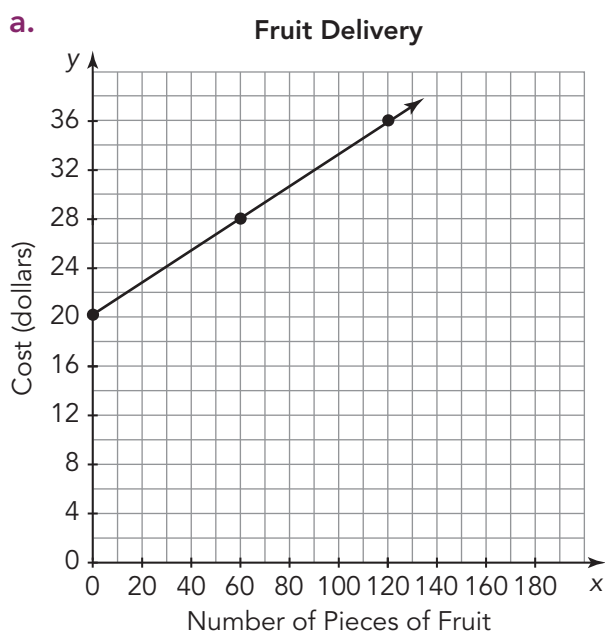
Describing Linear Equations

You have discovered that the equation $y = mx$ represents a proportional relationship. The equation represents every point (x, y) on the graph of a line with slope m that passes through the origin $(0, 0)$.
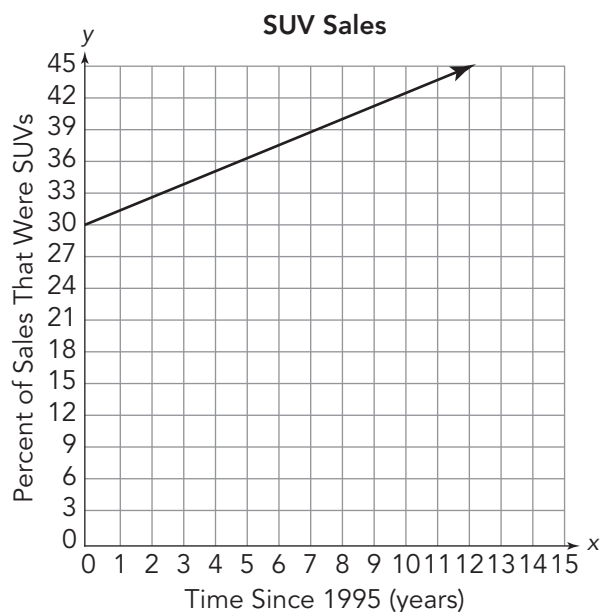
An equation of the form $y = mx + b$, where b is not equal to zero, represents a non-proportional relationship. This equation represents every point (x, y) on the graph of a line with slope m that passes through the point $(0, b)$.

1. **Consider each graph shown.**

   • **Determine whether the graph represents a proportional or non-proportional relationship.**

   • **Write an equation in the form $y = mx$ or $y = mx + b$ to represent the relationship between the independent and dependent quantities.**

a.

Fruit Delivery

Cost (dollars) vs Number of Pieces of Fruit

b.

Road Trip

Distance Traveled (miles) vs Time (hours)

c.

SUV Sales
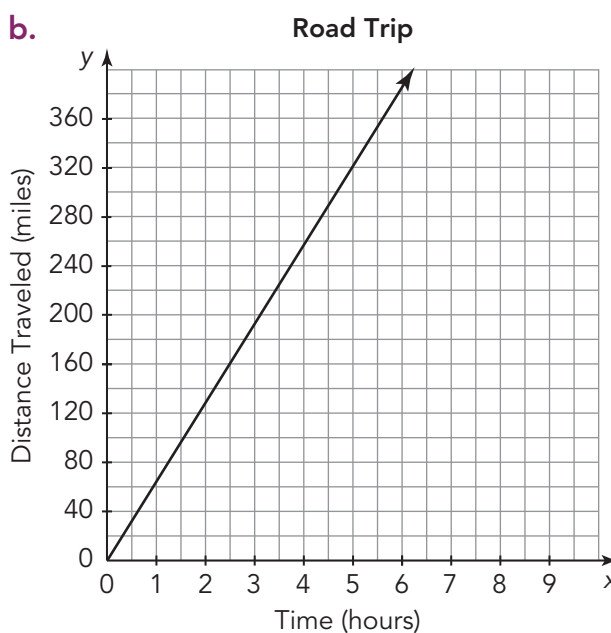
2. Determine the slope of this graph and write an equation to represent it. Describe a situation that could be modeled by this graph.
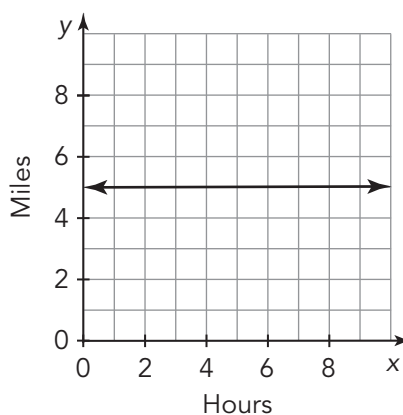
3. Complete the table of values to represent the linear relationship specified. Then, write an equation to represent the relationship.
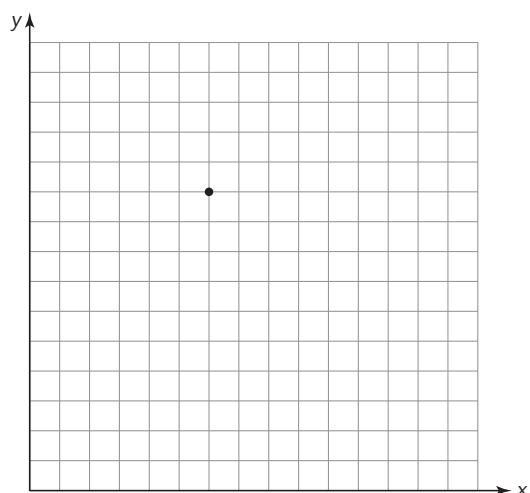
a. proportional relationship

x	y
0	
1	
2	12
3	
4	

b. non-proportional relationship

x	y
0	
1	
2	12
3	
4	

4. **Draw a line through the point and label the graph to represent the linear relationship specified. Then, write an equation.**

a. proportional relationship

b. non-proportional relationship
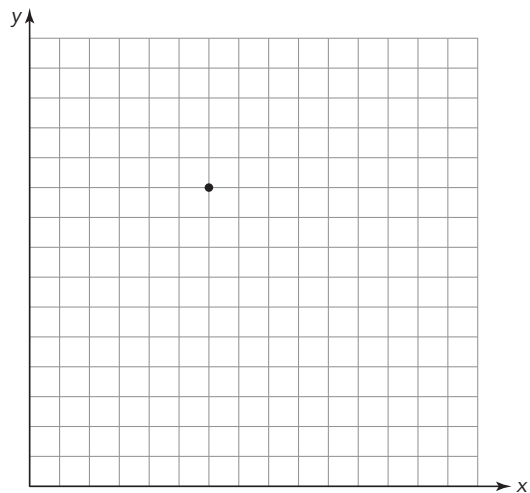
TALK the TALK 💬

A Web of Connections

In this lesson, you learned that the steepness of a line can be described by its slope, which is a concept that is connected to many other concepts you have learned previously.

1. Complete the graphic organizer to describe how steepness is related to slope, rate of change, unit rate, and the constant of proportionality. Include definitions, graphs, and equations. Be sure to address both proportional and non-proportional relationships.

SLOPE

RATE OF CHANGE

STEEPNESS
OF A
LINE

UNIT RATE

CONSTANT OF
PROPORTIONALITY
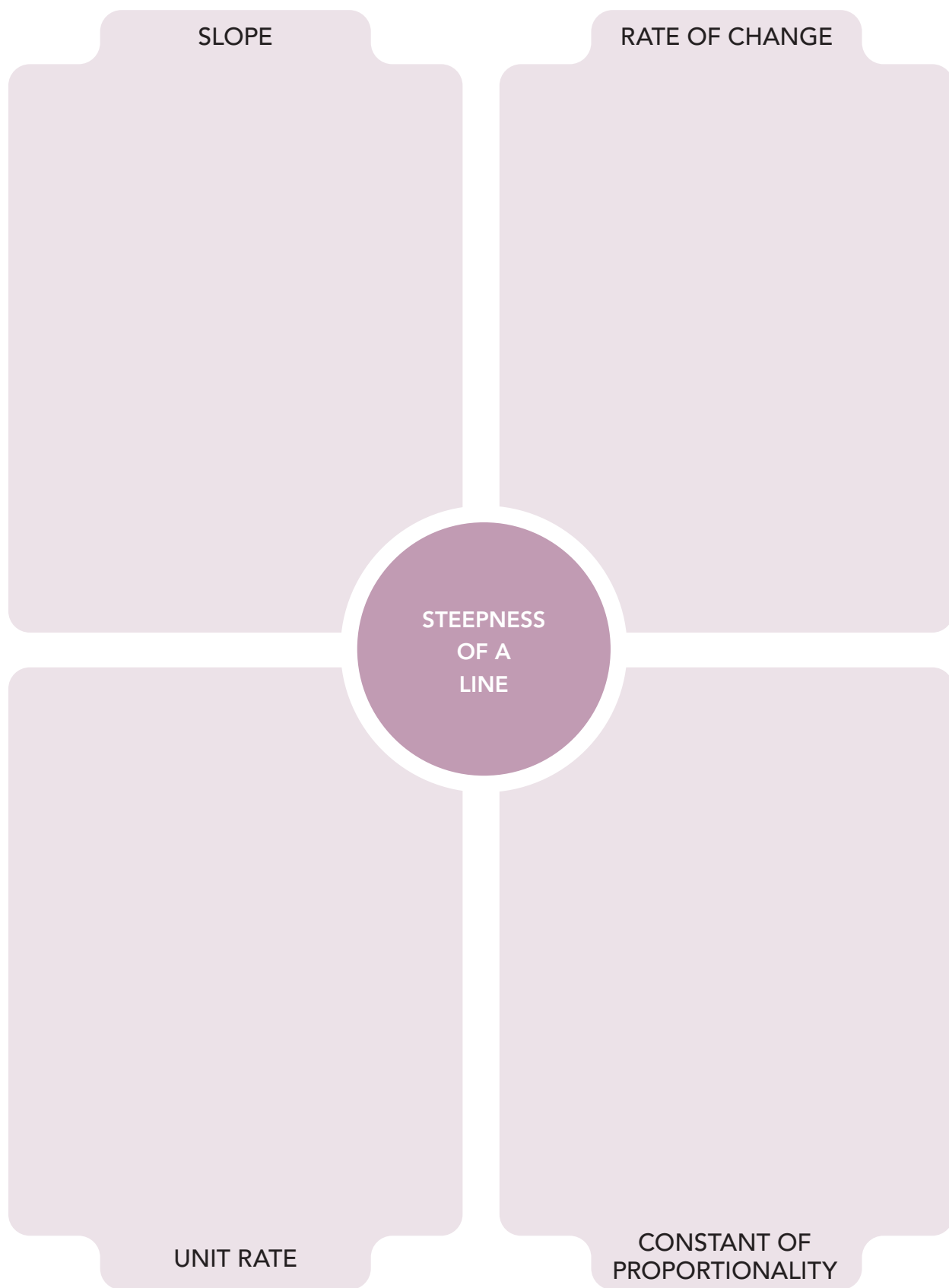
Assignment

Write

In your own words, explain how slope is related to the right triangles formed along the line. Use examples to illustrate your explanation.

Remember

- Slope is another name for the rate of change of a linear relationship graphed as a line.
- The equation for a proportional linear relationship is $y = mx$, where m is the slope. The equation represents all of the points (x, y) on the line.
- An equation for a non-proportional linear relationship is $y = mx + b$, where m is the slope and b is the y-coordinate of the point where the graph crosses the y-axis. The equation represents all of the points (x, y) on the line.

Practice

1. Maximilian is cleaning shrimp. He cleans 4 shrimp every minute. Use time in minutes as the independent quantity and the number of shrimp as the dependent quantity.
   a. Is the relationship proportional or non-proportional? Explain how you can determine this using a graph and the equation.
   b. Identify the unit rate of this relationship. Explain what the unit rate means in terms of the situation.
   c. Write an equation that determines the number of shrimp cleaned given any time.
   d. Create a graph of the relationship.

2. Consider each graph shown.
   - Determine whether the graph represents a proportional or non-proportional relationship.
   - Write an equation in the form $y = mx$ or $y = mx + b$ to represent the relationship between the independent and dependent quantities.
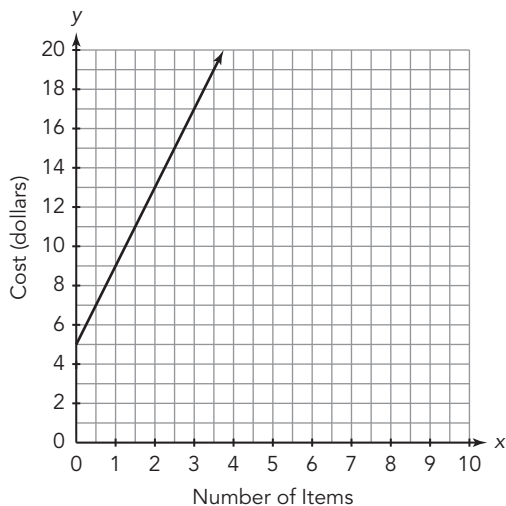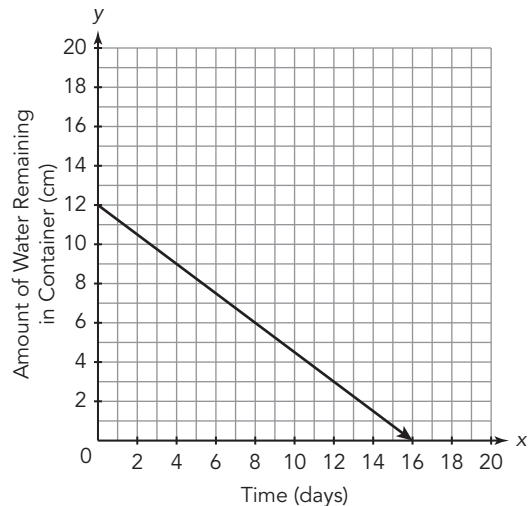
a.

b.

c.

Cost (dollars) vs. Number of Items

d.

Amount of Water Remaining in Container (cm) vs. Time (days)
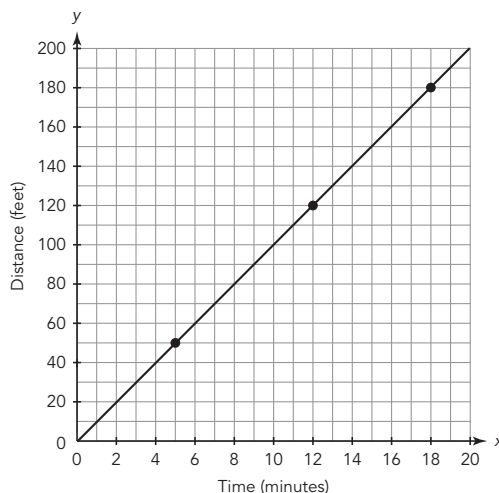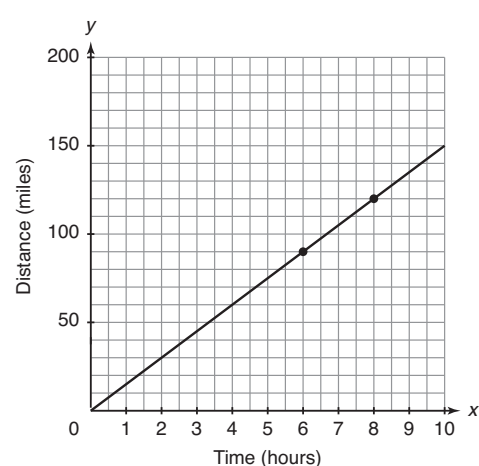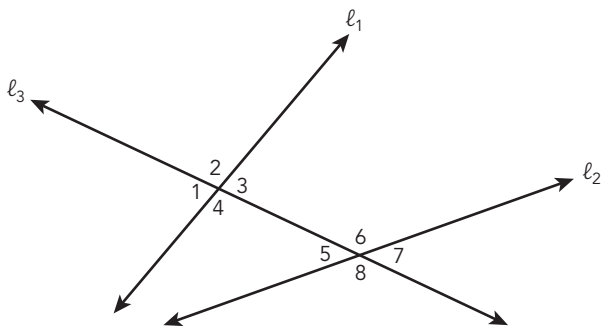
Stretch

Write an equation that determines where the graph crosses the y-axis, given the slope and the coordinates of one point.

Review

1. Determine whether each equation represents a proportional relationship.
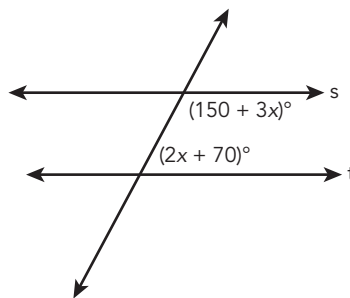   a. $y = 2.5x$
    b. $y = x - 4$

2. Examine the figure shown.

a. Name 2 pairs of same-side interior angles.
b. Name 2 pairs of congruent angles.
c. Name 2 pairs of supplementary angles.

3. In the diagram shown, line s and line t are parallel. Determine the measures of all the angles.
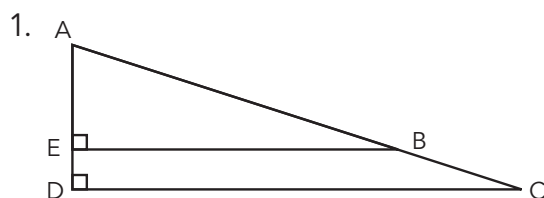

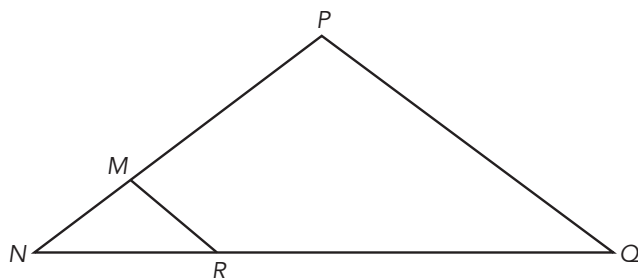$(150 + 3x)°$

$(2x + 70)°$

Slippery Slopes

3

Exploring Slopes Using Similar Triangles

WARM UP

For each diagram, describe how you can show that the triangles are similar.

1.

2. Given: *PQ* ‖ *MR.*

LEARNING GOALS

- Use similar triangles to show that the slope is the same between any two distinct points on a non-vertical line in a coordinate plane.
- Use right triangles to identify the slope of a line from a graph.

You have used similar triangles to describe the steepness of a line. How can you use similar triangles to explain why the slope is the same between any two distinct points on a non-vertical line?

Steep Grade

Consider the three street signs shown.

Discuss each question with your partner.

1. Where might you see each of the signs?

2. What do you know about the triangles on the signs?

**3. For the signs that include numbers, what do you think those numbers represent?**

Triangles and the Equation $y = mx$

In the previous lesson, *Jack and Jill Went Up the Hill,* you used patty paper to analyze the slope of the line $y = \frac{4}{3}x$ using similar triangles formed at $x = 1$, $x = 3$, and $x = 6$.

Now, let's investigate if the slope of a line is always the same between any two points on a line.

Consider the graph of $y = \frac{3}{2}x$.
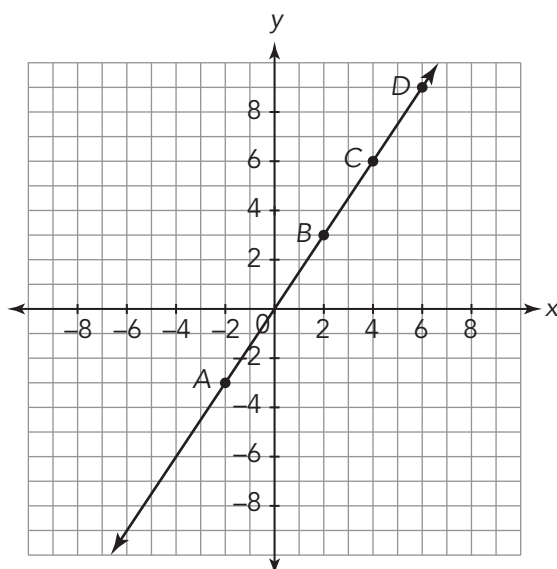
Remember, slope describes the direction and steepness of a line.

1. **Is the slope of the line positive or negative? Explain your reasoning.**

2. **Examine the slope between points *A* and *B*.**

   a. **Create a right triangle using points *A* and *B* and trace onto patty paper.**

   b. **Label the triangle with the vertical and horizontal distances.**

   c. **Label the patty paper with the slope of the line between points *A* and *B*.**

3. Does the orientation of the right triangle matter? Place your patty paper on the graph, use point *A* as the center of rotation, and rotate your triangle 180°.

   a. Compare and contrast these two triangles. How are they the same? How are they different?

   b. Does the new triangle give you the same slope? Explain your reasoning.

4. Create right triangles using points *B* and *C*, and then *B* and *D*.

   a. Label the horizontal and vertical distances.

   b. Label the patty paper with the slope of the line.

5. Compare the triangles created on the line. How can you verify that all of the triangles are similar?

6. What is the slope of the line?

7. Cooper claims that all right triangles formed on a given line are similar. Is Cooper correct? Explain your reasoning.

ACTIVITY

3.2

# Triangles, Slope, and the Equation $y = mx + b$

Consider the graph shown.
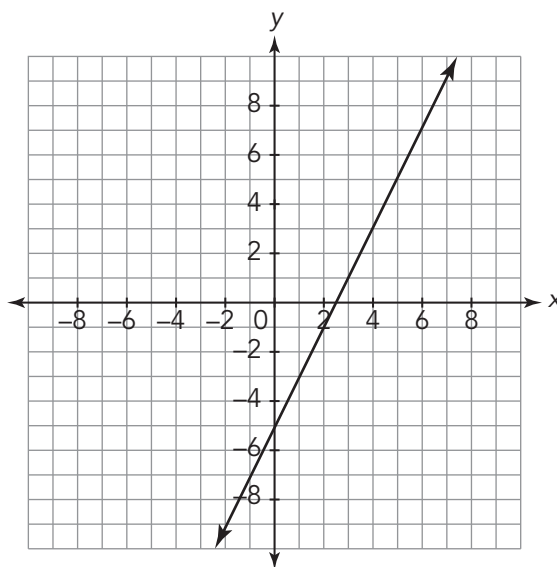
1. **Is the slope of the line positive or negative? Explain your reasoning.**

2. **Create at least three similar triangles using points on the line.**

   a. **Use any method to justify that these triangles are similar.**

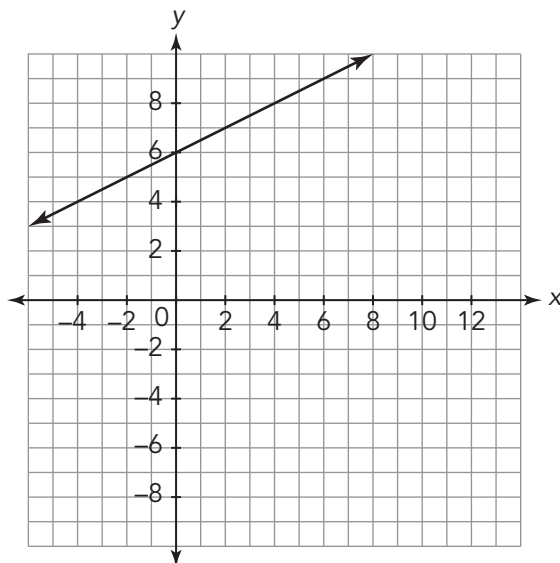   b. **Determine the slope of the line.**

3. How many similar triangles can be formed on the graph of a line? How do you know?

4. Consider each graph shown. Determine the slope of each line and then use similar triangles to justify that the slope is the same between any two points.

a.
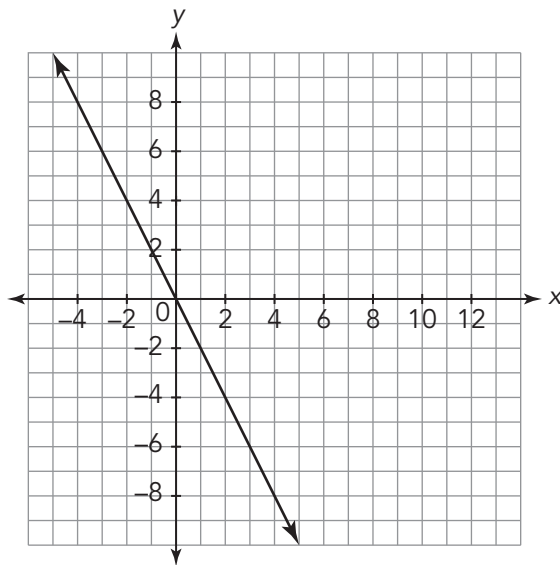
b.

TALK the TALK

Connecting Similar Triangles and Slope of a Line

Audra was absent for this lesson on the connection between similar triangles and the slope of a line. Write an explanation of what you learned in this lesson. Be sure to include how you can use a graph to determine the slope of a non-vertical line, and how you can use similar triangles to show the slope is the same between any two points on the line.
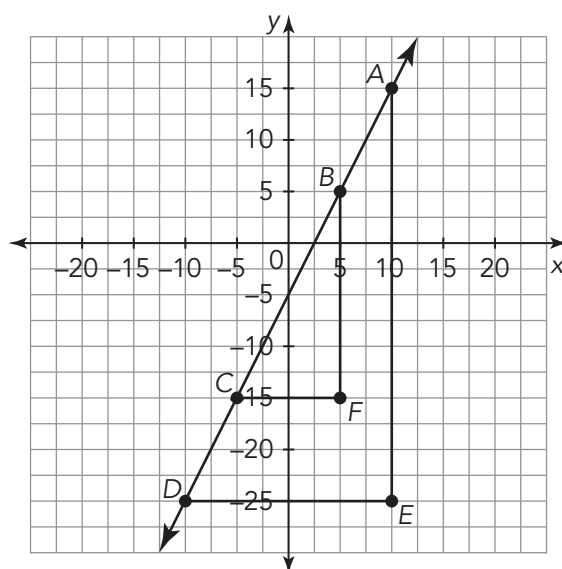
Assignment

Write

Explain why the slope between any two points on a line is always the same.

Remember

The properties of similar triangles can be used to explain why the slope m is the same between any two distinct points on a non-vertical line in the coordinate plane.

Practice

1. Consider the graph of the equation $y = 2x + 3$.

   a. The points on the line were used to create triangles. Describe the relationship between the two triangles.

   b. How can transformations be used to verify the relationship between the triangles?

   c. Use the similar triangles to determine the slope between any two points on the line.
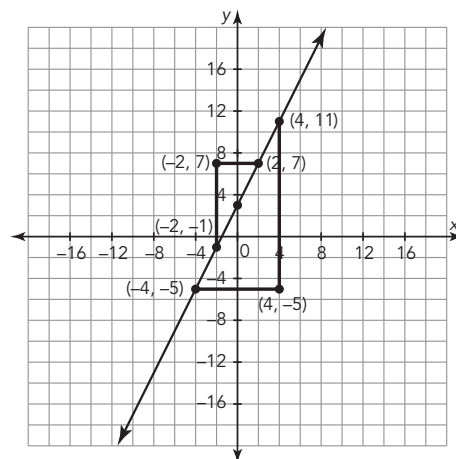
2. Consider each graph shown. Determine the slope of each line and then use similar triangles to justify that the slope is the same between any two points.
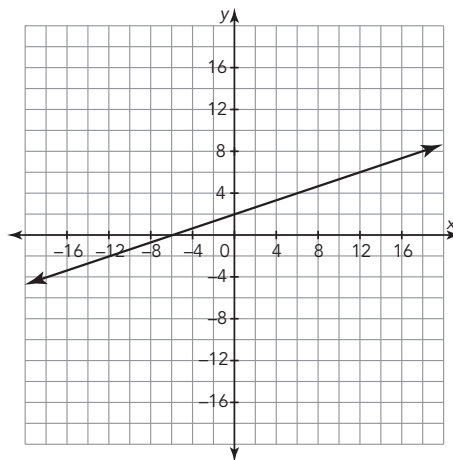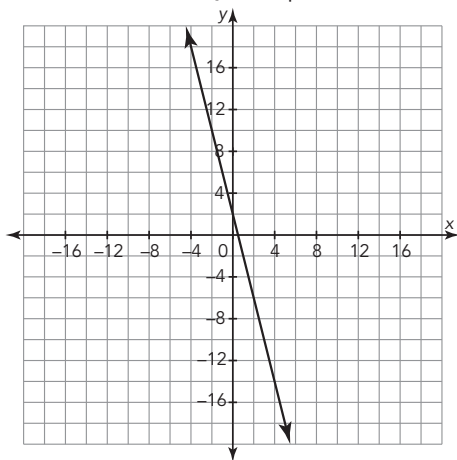
Stretch

Create a table of values for the equation $y = x^2$. Use the points with x-values of 0, 1, 2, and 3 to create triangles with the length of each base equal to 1 unit.

- Describe the relationship between the heights of the resulting triangles.
- Are the triangles similar? Explain your reasoning.

Review

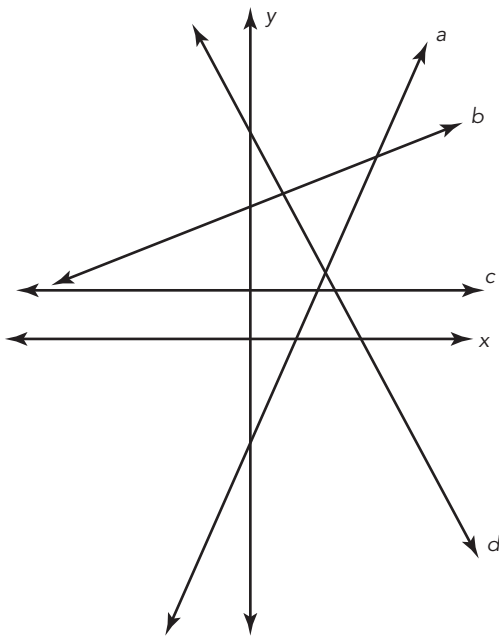1. Determine the unknown angle measure for each triangle.

   a. $m\angle A = 46°$, $m\angle B = 90°$, $m\angle C = ?$

   b. $m\angle P = ?$, $m\angle Q$: $10°$, $m\angle R = 110°$
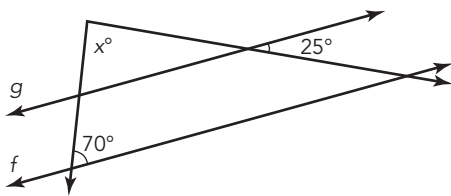
2. Consider the graph of lines a, b, c, and d.

   a. Which line(s) have positive slope?

   b. Which line(s) have negative slope?

3. Solve for the unknown angle measure given that $f \parallel g$.

Up, Down, and All Around

4

Transformations of Lines

WARM UP

Identify whether the equation represents a proportional or non-proportional relationship. Then state whether the graph of the line will increase or decrease from left to right.

1. $y = -2x - 9$

2. $y = 2x - 3$

3. $y = \frac{2}{3}x$

LEARNING GOALS

- Translate linear graphs horizontally and vertically.
- Use transformations to graph linear relationships.
- Determine the slopes of parallel lines.
- Identify parallel lines.
- Explore transformations of parallel lines.

You have learned how the coordinates of an image are affected when a pre-image is translated, reflected, rotated, or dilated. How can you use knowledge about geometric transformations to transform the graphs and equations of linear relationships?

Transformation Station

Consider $\triangle ABC$ with coordinates A (2, 2), B (8, 2), and C (8, 8) shown on the coordinate plane.

1. Suppose the triangle is translated in a single direction. In general, how does this affect the coordinates of the figure?
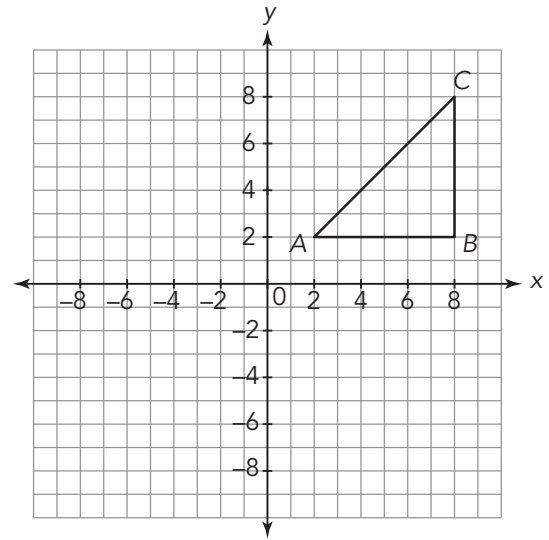
(x, y)	4 Units Up	4 Units Down	4 Units Left	4 Units Right
New Coordinates				

2. Suppose the triangle is reflected across an axis. How does this affect the coordinates of the figure?

(x, y)	x-Axis	y-Axis
New Coordinates		

3. Suppose the triangle is rotated through an angle with the origin as the center of rotation. How does this affect the coordinates of the figure?
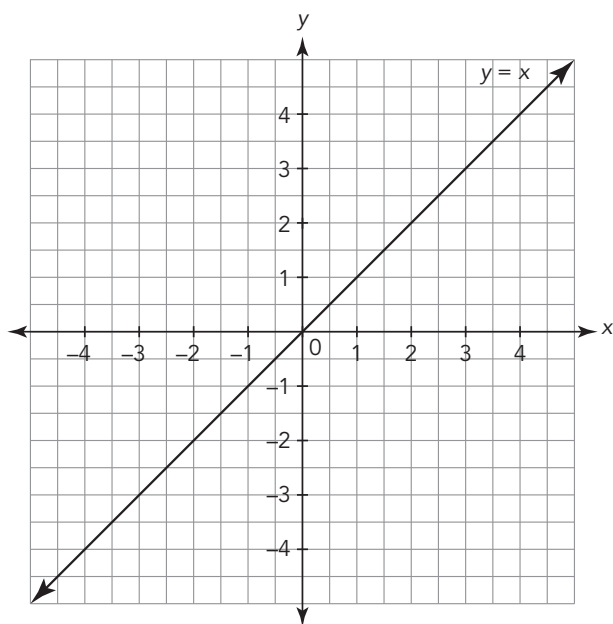
(x, y)	90° Counterclockwise	180°	270° Counterclockwise
New Coordinates			

4. Suppose the triangle is dilated by a factor of *m* with a center of dilation at the origin. How does this affect the coordinates of the figure?

(x, y)	Dilation
New Coordinates	

5. How do you think translations, reflections, rotations, and dilations affect lines?

In this activity, you will investigate how the equation of a line changes as you translate the line up and down the *y*-axis.

Consider the graph of the basic linear equation $y = x$, which is of the form $y = mx$. The line represents a proportional relationship with a rate of change, or slope, of 1.

1. **Trace the axes and the line $y = x$ on a sheet of patty paper.**

2. **Keep the *y*-axis on your patty paper on top of the corresponding *y*-axis of the coordinate plane. Slide the line $y = x$ up and down the *y*-axis.**

   a. **How does the slope of the line change as you move it up and down the y-axis?**

   b. **How do the coordinates of the line change as you move it up and down the y-axis?**

3. Translate the line $y = x$ up 4 units.

   a. Graph and label the line with its equation.

   b. Compare the equation of $y = x$ to the equation of its translation up 4 units. What do you notice?

4. Translate the line $y = x$ down 4 units.

   a. Graph and label the line with its equation.

   b. Compare the graph and equation of $y = x$ to the graph and equation of its translation down 4 units. What do you notice?

5. For any x-value, how does the y-value change when you translate $y = x$ up or down?

6. Are the translated lines proportional or non-proportional relationships? Explain your reasoning.
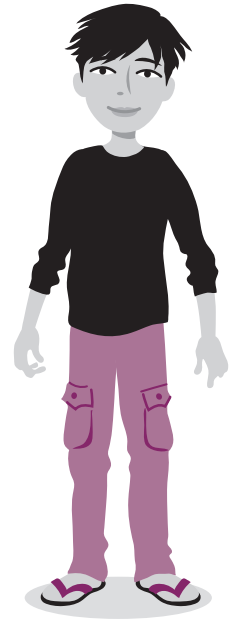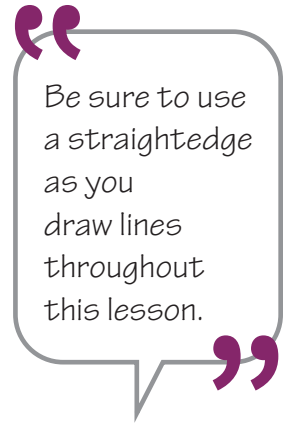
7. The lines on the graph are translations of the line represented by $y = x$.
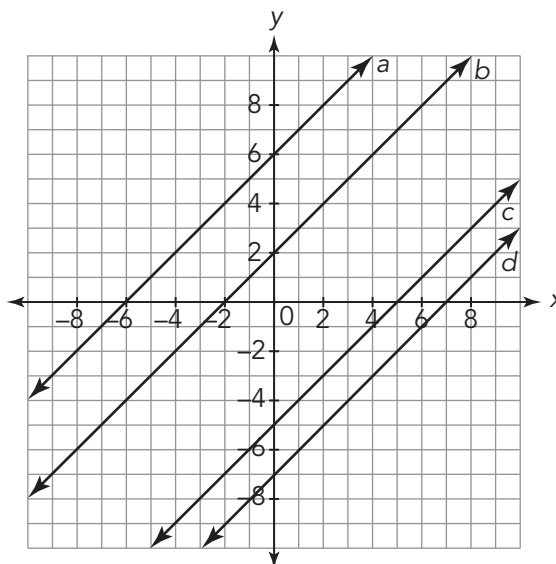
a. Describe each translation in terms of a translation up or down. Then write the equation.

b. Identify the slope of each line.

The lines drawn on the coordinate plane in Question 7 represent parallel lines. Remember that parallel lines are lines that lie in the same plane and do not intersect no matter how far they extend. Parallel lines are always equidistant.

8. Analyze the graph of each line and its corresponding equation.

a. How can you verify that the lines graphed are equidistant?

b. How can you tell by looking at the set of equations that the lines are parallel?
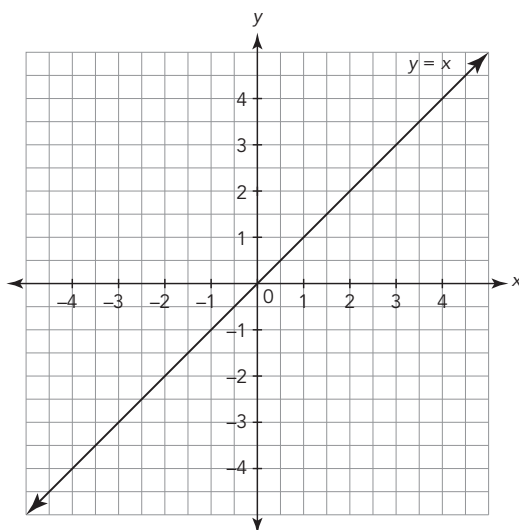
9. Based on your investigation, complete the sentence:

The line $y = x + b$ is a _____ of the line $y = x$

that maps the point (0, 0) onto the point _____ and maps the

point (1, 1) onto the point (1, _____).

The graph of the basic linear equation $y = x$ is shown on the coordinate plane.

Let's investigate how the line $y = x$ changes when the rate of change, or slope, changes.

1. **Use a thin piece of pasta to explore how the characteristics of the line change as you dilate the line $y = x$ to create the lines with equation $y = 2x$ and $y = \frac{1}{2}x$. Then complete the table based on your investigation.**

x	$y = x$	$y = 2x$	$y = \frac{1}{2}x$	$y = mx$
−2				
0				
1				
2				
4				

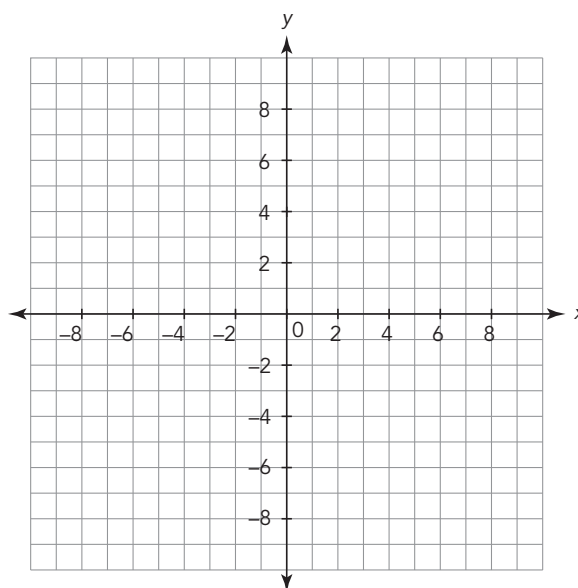2. **Based on your investigation, complete the sentence:**

   The line $y = mx$ is a _____ of the line

   $y = x$ that maps the point (0, 0) onto the point _____

   and maps the point (1, 1) onto the point (1, ____).

3. **Consider the equation $y = \frac{3}{4}x$. Use transformations to complete the table of values. Explain your strategy.**

x	$y = x$	$y = \frac{3}{4}x$
-2		
-1		
0		
1		
2		

4. **The equation $y = -x$ is a transformation of $y = x$.**

   a. **How are the equations similar? How are they different?**

b. Graph both equations to determine the transformation.

c. Based on your investigation, complete the sentence:

The line $y = -x$ is a _____ of the line $y = x$ that

maps the point (0, 0) onto the point _____ and maps

the point (1, 1) onto the point (1, _____).

> If you graph
> $y = 2x$,
> does your
> transformation
> still work to
> create the line
> $y = -2x$?

Using Transformations to Graph Lines

You have explored how the basic linear equation $y = x$ is translated to create the equation $y = x + b$ or dilated to create the equation $y = mx$. In this activity, you will combine both dilations and translations to graph equations of the form $y = mx + b$.

1. Consider the set of equations.

   • $y = 2x$

   • $y = 2x + 3$

   • $y = 2x - 5$

   • $y = 2x + 5$

   a. What do all of the equations have in common?
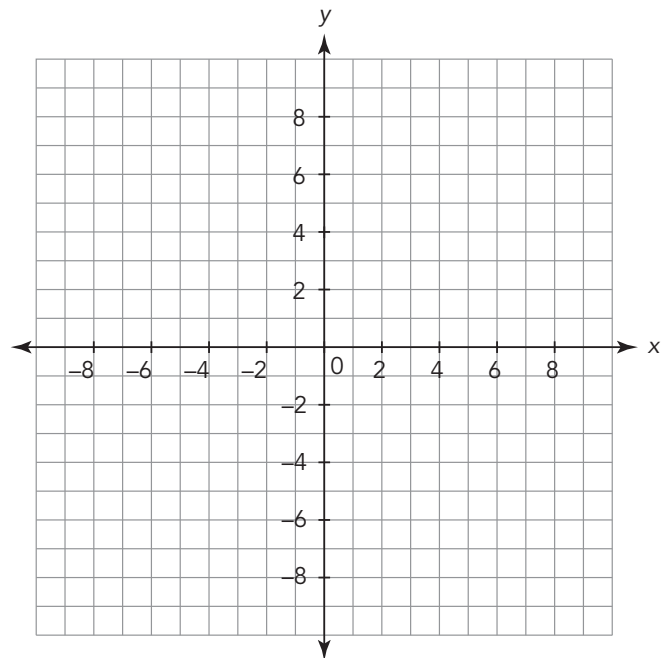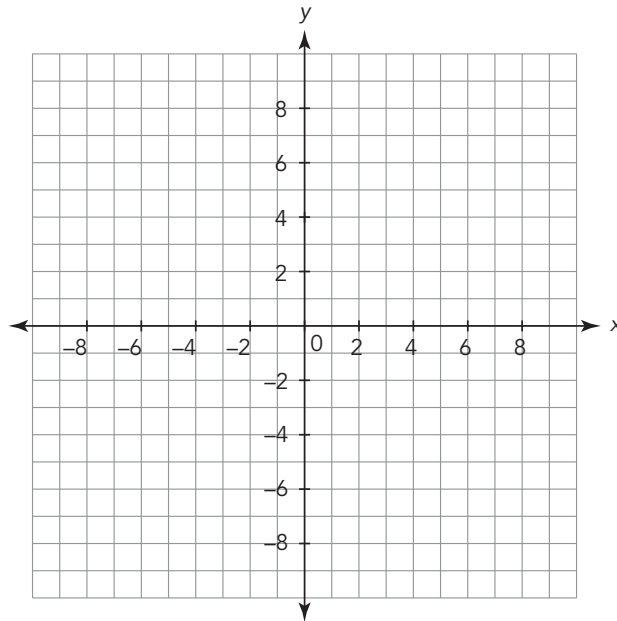

   b. Use transformations to graph each equation on the coordinate plane.

   c. Describe the relationship among the lines.

2. Consider the set of equations.

- $y = -3x$

- $y = -3x - 2$

- $y = -3x + 5$

- $y = -3x - 8$

a. What do all of the equations have in common?

b. Use transformations to graph each equation on the coordinate plane.

c. Describe the relationship among the lines.

d. Describe and use a strategy for verifying the relationship among the lines.
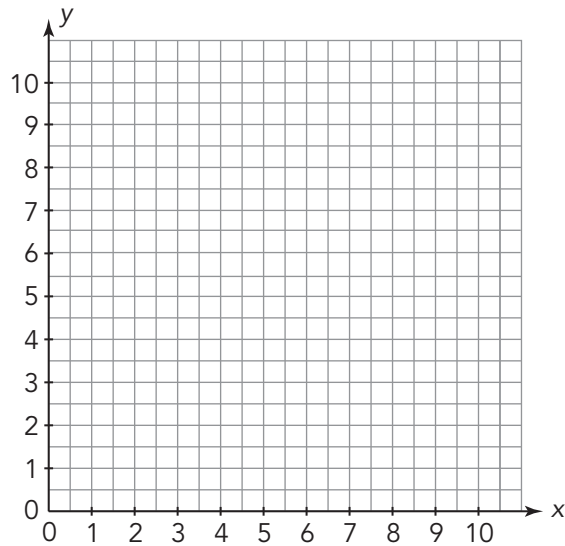
3. Consider these equations.

- $y = \frac{1}{2}x$

- $y = \frac{1}{2}x + 6$

- $y = \frac{1}{2}x - 3$

- $y = \frac{1}{2}x - 2$

   a. Without graphing, describe the graphical relationship among the lines.

   b. Explain how you determined the relationship.

4. Determine if the quadrilateral formed by joining the points A (3, 1), B (8, 1), C (10, 5), and D (5, 5) in alphabetical order is a parallelogram.

Now that you understand linear equations in terms of transformations, you can use transformations to graph lines.
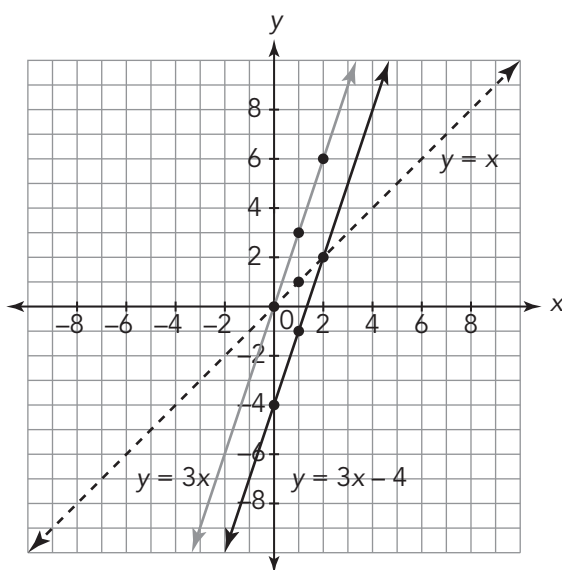
WORKED EXAMPLE

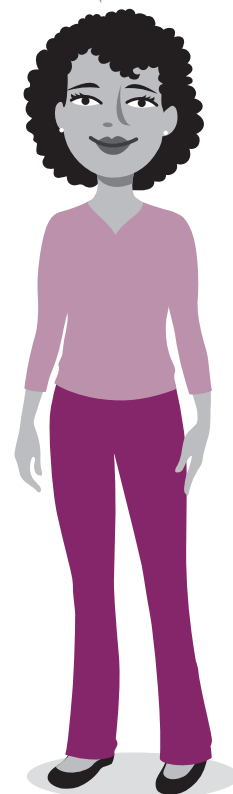Graph $y = 3x - 4$ using transformations of the basic linear equation $y = x$.

First, graph the basic equation, $y = x$, and consider at least 2 sets of ordered pairs on the line, for example (0, 0), (1, 1), and (2, 2).
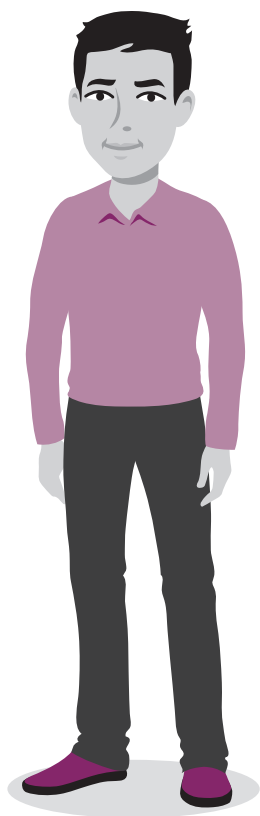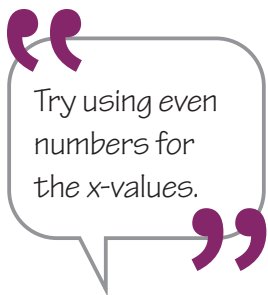
Then dilate the y-values by 3.
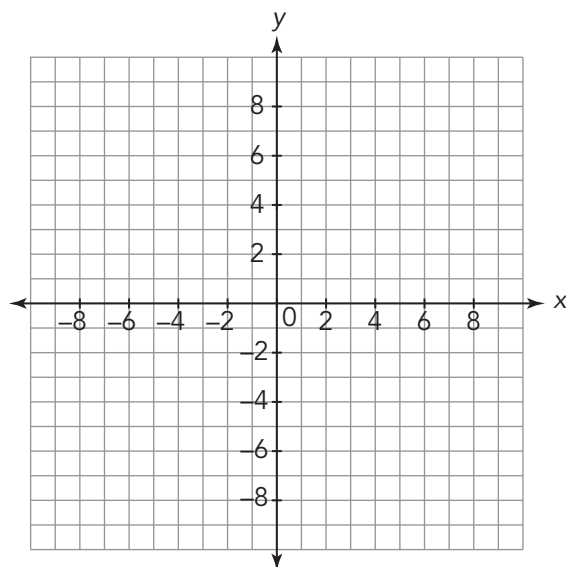
Finally, translate all y-values down 4 units.

> *Could you have translated the line $y = x$ down 4 units first, and then dilated the y-values by 3?*
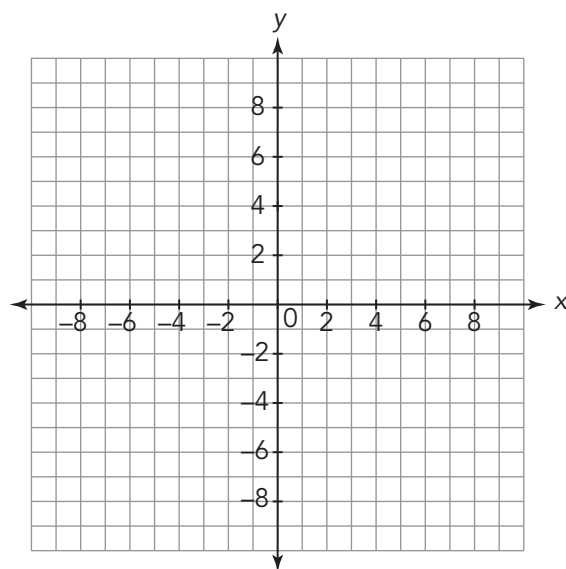
5. **Graph each equation using transformations. Specify which transformations you use.**

a. $y = \frac{1}{2}x + 5$
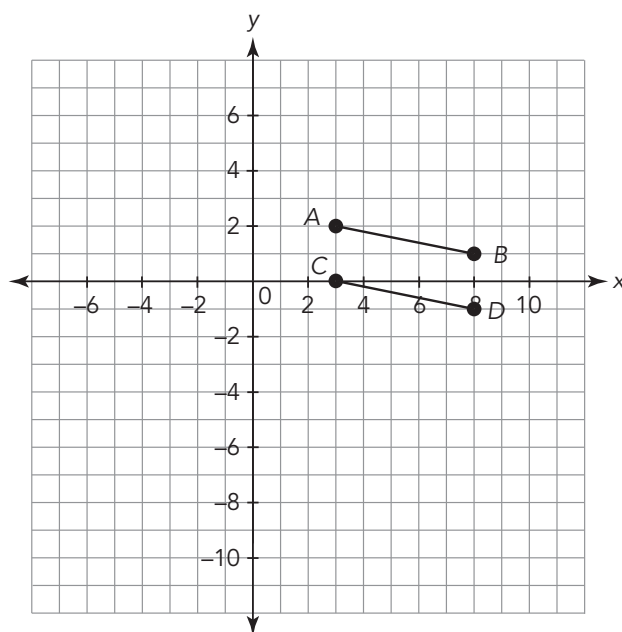

b. $y = \frac{3}{2}x - 3$

Reflecting and Rotating Parallel Lines

You have learned what happens when a line or figure is reflected across the *y*-axis. What happens if you reflect a pair of lines across the *y*-axis?

1. **Line segment *AB* and line segment *CD* are shown on the coordinate plane.**

> Patty paper might be helpful when reflecting across the *y*-axis.

   a. **What is the relationship between segments *AB* and *CD*? Justify your reasoning.**

   b. **Trace line segments *AB* and *CD* onto a sheet of patty paper. Reflect the line segments across the *y*-axis to create segments *A′B′* and *C′D′*.**

   c. **What are the coordinates of points *A′*, *B′*, *C′*, and *D′*?**

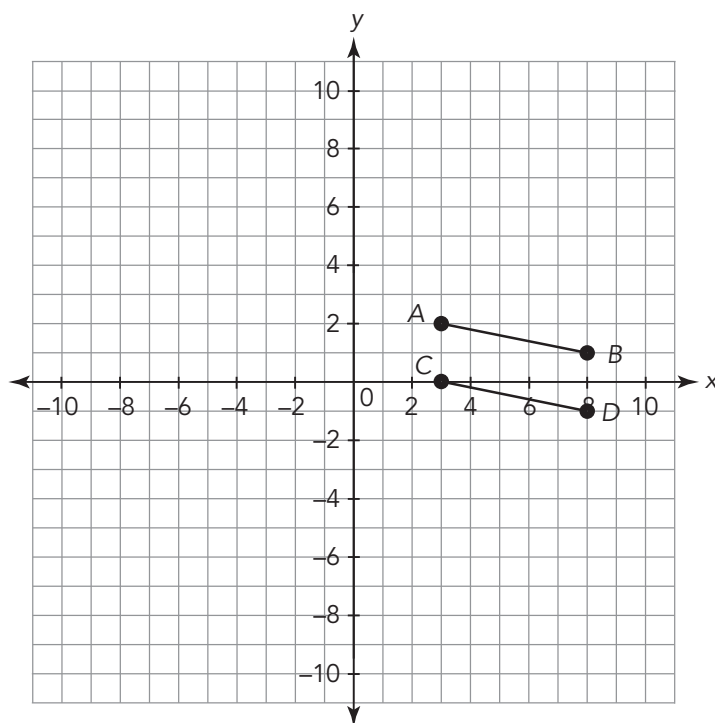   d. **What is the relationship between segments *A′B′* and *C′D′*? Justify your reasoning.**

e. Extend segments *AB, CD, A'B',* and *C'D'* to create lines *AB, CD, A'B',* and *C'D'*. Draw the lines on your graph. What do you notice about the relationship between the lines?

f. Reflecting parallel lines across the same line of reflection results in lines that are _____.

Let's explore what happens when the segments and lines created from the points *A* (3, 2), *B* (8, 1), *C* (3, 0), and *D* (8, −1) are rotated.

2. Consider the line segments *AB* and *CD* as shown on the coordinate plane.

You learned that when you rotate a point (x, y) 90 degrees counterclockwise about the origin, the image of the point is $(-y, x)$. However, what happens when you rotate two parallel lines?


a. Rotate each point 90° counterclockwise to create segments *A'B'* and *C'D'*. What are the coordinates of points *A', B', C',* and *D'*?

b. What is the relationship between line segments *A'B'* and *C'D'*? Justify your reasoning.

c. Rotate each original point 180° to create a new set of segments. What are the coordinates of the new points?

d. What is the relationship between the segments created by rotating the original points 180°? Justify your reasoning.

e. Extend line segments *AB*, *CD*, *A'B'*, and *C'D'* to create lines AB, CD, A'B', and C'D'. Draw the lines on graph. What do you notice about the relationship between the lines?

f. Rotating parallel lines results in lines that are _____.

TALK the TALK 💬

Are They Parallel?

1. Which transformations of linear graphs result in parallel lines? Explain each response.

    a. dilation by a non-zero factor other than 1

    b. translation up or down

    c. reflection across an axis

    d. rotation 90° counterclockwise

2. Create and graph four linear equations that represent lines with the same slope. Label each line with its corresponding equation.
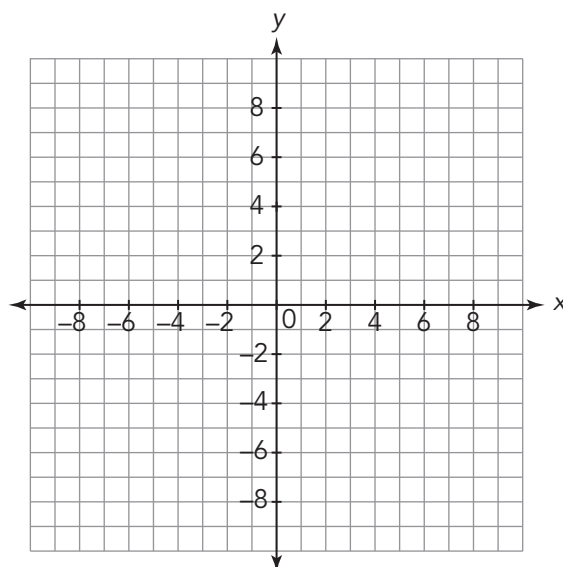
Write

Explain how to use transformations of the basic equation $y = x$ to graph the equation $y = mx + b$.

Remember

Translations, reflections, and rotations map parallel lines and line segments to corresponding parallel lines and line segments.

Practice

1. Write an equation for each linear relationship after transforming $y = x$.

   a. dilation by a factor of $\frac{5}{6}$

   b. dilation by a factor of 8

   c. reflection across the x-axis

   d. translation down 6 units

   e. dilation by a factor of 2, then a translation up 3 units

   f. reflection across the x-axis, dilation by a factor of 3, and then a translation down 9 units

2. Use the graph of the linear relationship shown to complete each task.

   a. Write the equation of the line.

   b. Write the equation of the line after a translation down 8 units. Graph the line.

   c. Write the equation of the line after a translation up 8 units. Graph the line.
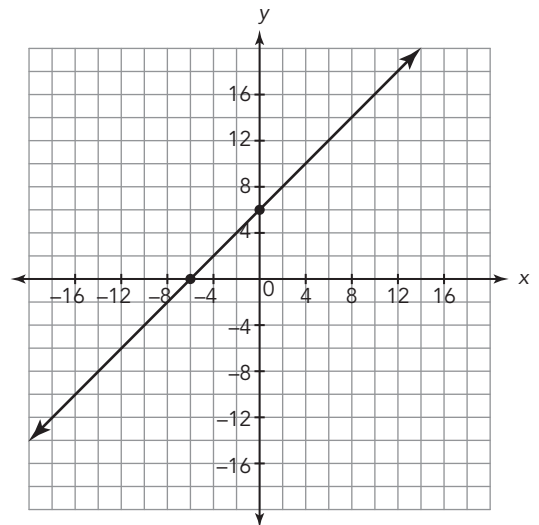

Stretch

Graph each given sequence of transformations. Are the equations the same? Explain why the equations must be the same or why they are not the same. Use transformations to support your answer.
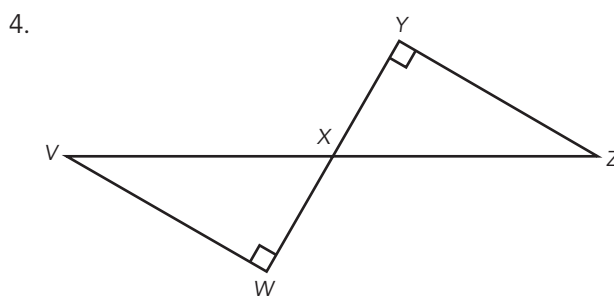
1. Translate $y = x$ up 4 units, and then dilate by a factor of 2.
2. Dilate $y = x$ by a factor of 2, and then translate up 4 units.

Review

Draw similar triangles on the graph to determine each slope.

1.

2.
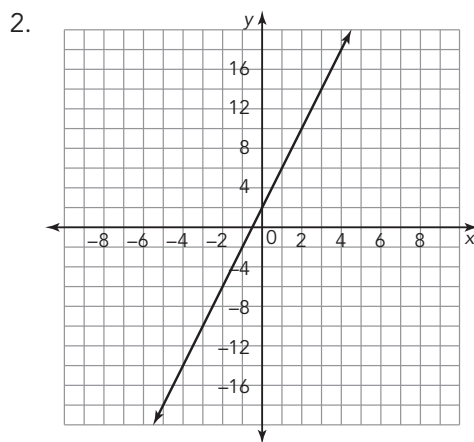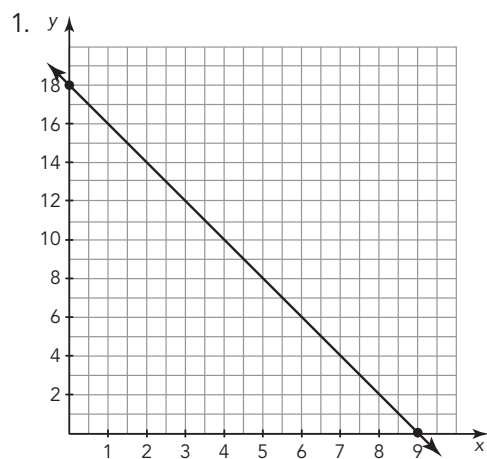
Identify the similar triangles and explain how the triangles are similar by the Angle-Angle Similarity Theorem.

3.

4.
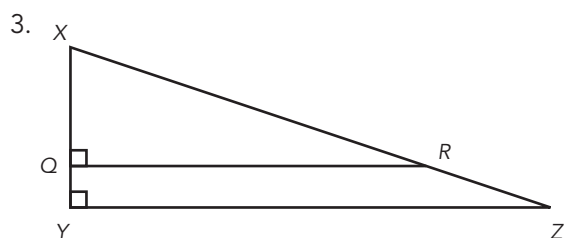
Solve each proportion for the unknown.

5. $\frac{2}{3} = \frac{x}{3.5}$

6. $\frac{0.6}{m} = \frac{4}{30}$

From Proportions to Linear Relationships Summary

KEY TERMS

- proportional relationship
- constant of proportionality
- rate of change
- slope

LESSON 1 — **Post-Secondary Proportions**

A **proportional relationship** is one in which the ratio of the inputs to the outputs is constant. For example, the ratio of women to men at a certain university is 3 : 2.

Proportional relationships can be represented using tables, graphs, and equations.

In a table, the values in a proportional relationship increase or decrease at a constant rate beginning or ending at (0, 0). The 3 : 2 ratio of women to men at a university is represented by this table.

Female Students Enrolled in a University	Male Students Enrolled in a University
0	0
600	400
1500	1000

On a graph, a proportional relationship is represented as a linear graph passing through the origin. The given graph shows the same proportional relationship as represented by the table.

The equation for a proportional relationship is written in the form $y = kx$, where x represents an input value, y represents an output value, and k represents some constant that is not equal to 0. The constant k is called the **constant of proportionality.** The 3 : 2 ratio of women to men at a university is represented by the equation $y = \frac{2}{3}x$. The constant of proportionality is $\frac{2}{3}$.

Male Students Enrolled (hundreds) vs *Female Students Enrolled (hundreds)*
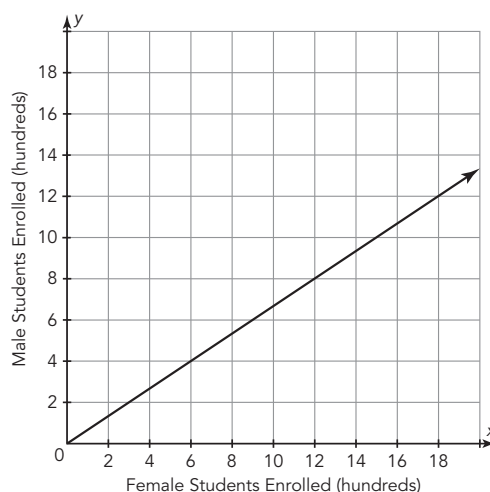
Jack and Jill Went Up the Hill

The **rate of change** for a situation is the amount that the dependent quantity changes compared with the amount that the independent quantity changes.

In any linear relationship, **slope** describes the direction and steepness of a line and is usually represented by the variable m. Slope is another name for the rate of change of a linear relationship graphed as a line. The slope of the line is constant between any two points on the line. The sign of the slope indicates the direction of a line. If the slope of a line is positive, then the graph will increase from left to right. If the slope of a line is negative, then the graph will decrease from left to right.
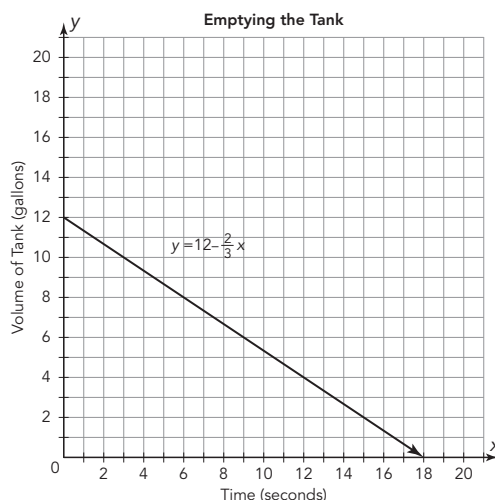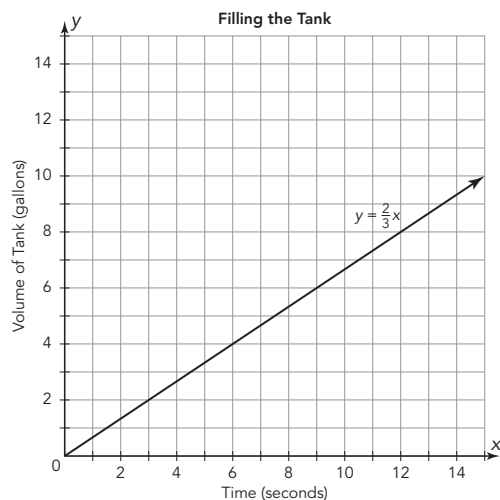
The equation $y = mx$ represents a proportional relationship. The equation represents every point (x, y) on the graph of a line with slope m that passes through the origin $(0, 0)$.

An equation of the form $y = mx + b$, where b is not equal to 0, represents a non-proportional relationship. This equation represents every point (x, y) on the graph of a line with slope m that passes through the point $(0, b)$. For example, the graph shown represents a non-proportional relationship where $m = 2$ and $b = 4$.
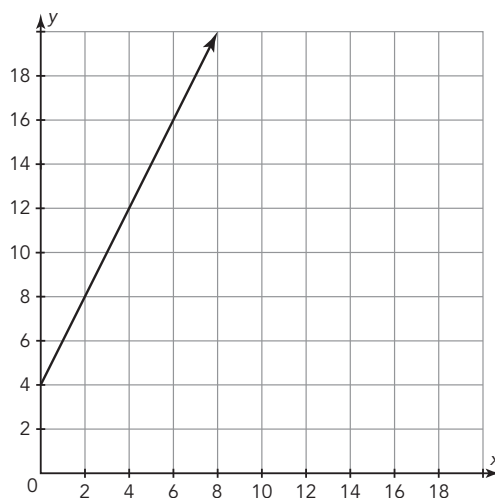
A line with a negative slope goes in the opposite direction. It decreases from left to right.

For example, consider the two graphs. The first represents a tank being filled at $\frac{2}{3}$ gallon per second. The second represents the tank being emptied at $\frac{2}{3}$ gallon per second, starting at 12 gallons.

Filling the Tank — Volume of Tank (gallons) vs Time (seconds), $y = \frac{2}{3}x$

Emptying the Tank — Volume of Tank (gallons) vs Time (seconds), $y = 12 - \frac{2}{3}x$

The slope of the line representing the tank being filled is $\frac{2}{3}$. You can draw a triangle to represent the slope of this line and then horizontally reflect it onto the line representing the tank being emptied. This shows that the slope of this line is $-\frac{2}{3}$.


LESSON

3

Slippery Slopes

The properties of similar triangles can be used to explain why the slope *m* is the same between any two distinct points on a non-vertical line on the coordinate plane.

For example, Points *A, B, D,* and *E* along the graphed line can be used to create two right triangles in the coordinate plane.

Because ∠*BAC* and ∠*EDF* are corresponding angles on parallel lines cut by a transversal, you know that ∠*BAC* ≅ ∠*EDF*. Likewise, because ∠*ABC* and ∠*DEF* are corresponding angles on parallel lines cut by a transversal, you know that ∠*ABC* ≅ ∠*DEF*. Therefore, by the AA Similarity Theorem, Δ*ABC* is similar to Δ*DEF*.
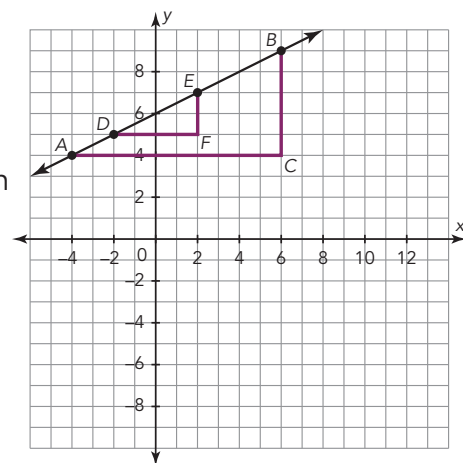
In both triangles, the ratio of the vertical distance and the horizontal distance is $\frac{1}{2}$. The slope of the line is the same between points A and B and between points D and E.

Translations, reflections, and rotations map parallel lines and line segments to corresponding parallel lines and line segments.

The line $y = x + b$ is a translation of the line $y = x$ that maps the point (0, 0) to the point (0, b) and maps the point (1, 1) to the point (1, 1 + b).

Up, Down, All Around

The line $y = mx$ is a dilation of the line $y = x$, which maps the point (0, 0) to the point (0, 0) and maps the point (1, 1) to the point (1, m).

The line $y = -x$ is a reflection of the line $y = x$, which maps the point (0, 0) to the point (0, 0) and maps the point (1, 1) to the point (1, −1).

For example, you can graph $y = 3x - 4$ using transformations of the basic linear equation $y = x$.

First, graph the basic equation $y = x$, and consider at least two sets of ordered pairs on the line, for example (0, 0), (1, 1), and (2, 2).

Then dilate the y-values by 3.

Finally, translate all y-values down 4 units.